

THE CLASSIFICATION OF MINIMAL PRODUCT-QUOTIENT SURFACES WITH $p_g = 0$.

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This article is dedicated to the memory of our dear friend and collaborator Fritz Grunewald

INTRODUCTION

The present article is the fourth in a series of papers (cf. [BC04], [BCG08], [BCGP08]), where the goal is to contribute to the classification problem of surfaces of general type by giving a systematic way to construct and distinguish algebraic surfaces.

We will use the basic notations from the classification theory of complex projective surfaces, in particular the basic numerical invariants K_S^2 , $p_g := h^0(S, \Omega_S^2)$, $q(S) := h^1(S, \mathcal{O}_S)$; the reader unfamiliar with these may consult e.g. [Be83].

The methods we introduced in the above cited articles, and substantially develop and refine in the present paper are in principle applicable to many more situations. Still we restrict ourselves to the case of surfaces of general type with geometric genus $p_g = 0$.

It is nowadays well known that minimal surfaces of general type with $p_g = 0$ yield a finite number of irreducible components of the moduli space of surfaces of general type. Although it is theoretically possible to describe all irreducible components of the moduli space corresponding to surfaces of general type with $p_g = 0$, this ultimate goal is far out of reach, even if there has been a substantial progress in the study of these surfaces especially in the last five years. We refer to [BCGP08] and [BCP10] for a historical account and recent update on what is known about surfaces of general type with $p_g = 0$.

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We study the following situation: let G be a finite group acting on two compact Riemann surfaces C_1, C_2 of respective genera at least 2. We shall consider the diagonal action of G on $C_1 \times C_2$ and in this situation we say for short: the action of G on $C_1 \times C_2$ is *unmixed*. By [Cat00] we may assume wlog that G acts faithfully on both factors.

Definition 0.1. *The minimal resolution S of the singularities of $X = (C_1 \times C_2)/G$, where G is a finite group with an unmixed action on the direct product of two compact Riemann surfaces C_1, C_2 of respective genera at least two, is called a product-quotient surface.*

X is called the quotient model of the product-quotient surface.

Remark 0.2. 1) It is possible that two product-quotient surfaces with different quotient models are birational or even isomorphic. By a slight abuse of notation, we still call $X = (C_1 \times C_2)/G$ "the" quotient model of the product-quotient surface, because we use this notation only if we have fixed two Riemann surfaces and the action of a finite group on each of them.

2) If X has "mild" (to be precise: at most canonical) singularities, then the quotient model X is equal to the canonical model of the product-quotient surface S , which is unique. For the definition of canonical surface singularities (which are also called rational double points) we refer to [Mat02], def. 4-2-1, thm. 4-6-7.

3) In the general setting, i.e., if X has non canonical singularities, S is not necessarily a minimal surface, i.e., it may contain smooth rational curves with selfintersection -1 . This is a substantial obstacle we have to overcome in the present paper. We profit from the fact that the construction of our surfaces is quite explicit (cf. section 5).

In general there is no way to determine rational curves on surfaces of general type. In fact, a famous still unsolved conjecture by S. Lang asserts that a surface of general type can contain only a finite number of rational curves.

The systematic classification of product-quotient surfaces with $p_g = 0$ was started and carried through in [BC04], [BCG08], [BCGP08] for all surfaces whose canonical model is equal to $(C_1 \times C_2)/G$.

[BC04] classifies the surfaces $X = (C_1 \times C_2)/G$ with G being an abelian group acting freely and $p_g(X) = 0$. This classification is extended in [BCG08] to the case of an arbitrary group G . We want to point out that in the first paper all calculations were done by hand, whereas in the second one the computations could not be done by hand, but they were still "computer aided hand calculations".

In [BCGP08] instead we dropped the assumption that G acts freely on $C_1 \times C_2$, we classified product-quotient surfaces with $p_g = 0$ whose quotient model is indeed the canonical model (i.e., X has at most canonical singularities, note that here

automatically $K_S^2 > 0$). In this case for the first time a systematic use of a computer algebra program was strictly needed in order to obtain a complete classification.

In the present paper we drop any restriction on the singularities of X . We succeed to give a complete classification of product-quotient surfaces S with $K_S^2 > 0$.

In order to obtain this result we had to substantially refine our previous MAGMA code, and for the first time we encountered serious problems of complexity and memory usage. Especially, as K_S^2 gets smaller (≤ 0), the problem of finding the possible singular locus of X gets more and more time and memory demanding. In order to finish the classification of product quotient surfaces with $p_g = 0$ one has to deal not only with the above mentioned computational problems, but also with the problem of bounding the number of rational curves on S , which in view of the previously mentioned Lang's conjecture is foreseen to be hard.

We are interested in the minimal model of the constructed surfaces, in order to locate them in the geography of the fine classification of the surfaces of general type. We determined the minimal model of all these surfaces; the last two sections are dedicated to this scope. It turns out that all except one are in fact minimal. We call this last surface *the fake Godeaux surface*, because a minimal surface with the same invariants p_g and K^2 is called *numerical Godeaux surfaces*. The section 5 is dedicated to it.

The following summarizes the results of the series of four papers.

- Theorem 0.3.** (1) *Surfaces S isogenous to a product (i.e., S is an étale quotient of a product of two compact Riemann surfaces of respective genera at least 2 by a finite group) with $p_g(S) = q(S) = 0$ form 17 irreducible connected components of the moduli space \mathfrak{M} of surfaces of general type. Exactly 13 of these families are families of product-quotient surfaces.*
- (2) *Minimal product-quotient surfaces with $p_g = 0$ form exactly 72 irreducible families.*
- (3) *There is exactly one product-quotient surface with $K_S^2 > 0$ which is non minimal. It has $K_S^2 = 1$, $\pi_1(S) = \mathbb{Z}/6\mathbb{Z}$ and its minimal model has $K^2 = 3$.*

Remark 0.4. 1) Part 1 is proved in [BC04], [BCG08].

2) Of the 72 families of part 2, 40 are constructed in [BC04], [BCG08], [BCGP08]. The remaining 32 families, as well as the fake Godeaux surface are new, and come out from our main classification result here.

Therefore we contribute to the existing knowledge about the complex projective surfaces S of general type with $p_g(S) = 0$ and their moduli spaces, constructing 33 new families of such surfaces realizing 14 hitherto unknown topological types.

The product-quotient surfaces mentioned in part 2 of the above theorem are listed in tables 1 and 2. We list the following information in the columns of the tables:

- Sing X is given as a sequence of rational numbers with multiplicities, describing the types of the cyclic quotient singularities, e.g., $2/3^2$ means 2 singular points of type $\frac{1}{3}(1, 2)$;
- N is the number of irreducible families; indeed our tables have only 60 lines, but we collect in the same line N families, which share all the other data; the number of lines, counted with multiplicity N is 72 (the number of families of theorem 0.3, 2));
- K_S^2 is the selfintersection of the canonical divisor, G the group, H_1 is the homology, and π_1 is the fundamental group.
- t_1, t_2 are the signatures of the corresponding polygonal groups, cf. definition 0.8 and the subsequent discussion.

For the groups occurring in tables 1, 2, we use the following notation: we denote by \mathbb{Z}_d the cyclic group of order d , \mathfrak{S}_n is the symmetric group in n letters, \mathfrak{A}_n is the alternating group and Q_8 is the quaternion group of order 8.

$PSL(2, 7)$ is the group of 2×2 matrices over \mathbb{F}_7 with determinant 1 modulo the subgroup generated by $-Id$.

$D_{p,q,r} = \langle x, y | x^p, y^q, xyx^{-1}y^{-r} \rangle$, and $D_n = D_{2,n,-1}$ is the usual dihedral group of order $2n$.

$G(n, k)$ for instance is the k -th group of order n in the MAGMA database of small groups.

In the sequel we shall give some consequences of the above theorem:

Comparing tables 1 and 2 with the constructions existing in the literature, as listed in table 1 of [BCP10], we note

Corollary 0.5. *Minimal surfaces of general type with $p_g = q = 0$ and with $3 \leq K^2 \leq 6$ realize at least 45 topological types.*

Note that before proving the results summarized in theorem 0.3 only 12 topological types of surfaces of general type with $p_g = q = 0$ and with $3 \leq K^2 \leq 6$ were known. In 2010 Cartwright and Steger (cf. [CaST10]) constructed 11 surfaces with $K_S^2 = 3$ and new mutually different fundamental groups, see [BCP10], especially table 1, for a more precise account on what was previously known in the literature.

In the present paper we construct 13 surfaces with new topological types (two of them were independently found by Cartwright and Steger).

The biggest impact on the "zoo" of surfaces of general type with $p_g = 0$ of our work is the case $K_S^2 = 5$: here we raise the number of known different topological types from one to seven.

Surfaces with $p_g = 0$ are also very interesting in view of Bloch's conjecture ([Blo75]), predicting that for surfaces with $p_g = 0$ the group of zero cycles modulo rational equivalence is isomorphic to \mathbb{Z} .

TABLE 1. Minimal product-quotient surfaces of general type with $p_g = 0$, $K^2 \geq 4$

| K_S^2 | Sing X | t_1 | t_2 | G | N | $H_1(S, \mathbb{Z})$ | $\pi_1(S)$ |
|---------|-------------|------------|------------|---|---|--|---|
| 8 | \emptyset | $2, 5^2$ | 3^4 | \mathfrak{A}_5 | 1 | $\mathbb{Z}_3^2 \times \mathbb{Z}_{15}$ | $1 \rightarrow \Pi_{21} \times \Pi_4 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | 5^3 | $2^3, 3$ | \mathfrak{A}_5 | 1 | \mathbb{Z}_{10}^2 | $1 \rightarrow \Pi_6 \times \Pi_{13} \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | $3^2, 5$ | 2^5 | \mathfrak{A}_5 | 1 | $\mathbb{Z}_2^3 \times \mathbb{Z}_6$ | $1 \rightarrow \Pi_{16} \times \Pi_5 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | $2, 4, 6$ | 2^6 | $\mathfrak{S}_4 \times \mathbb{Z}_2$ | 1 | $\mathbb{Z}_2^4 \times \mathbb{Z}_4$ | $1 \rightarrow \Pi_{25} \times \Pi_3 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | $2^2, 4^2$ | $2^3, 4$ | $G(32, 27)$ | 1 | $\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$ | $1 \rightarrow \Pi_5 \times \Pi_9 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | 5^3 | 5^3 | \mathbb{Z}_5^2 | 2 | \mathbb{Z}_5^2 | $1 \rightarrow \Pi_6 \times \Pi_6 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | $3, 4^2$ | 2^6 | \mathfrak{S}_4 | 1 | $\mathbb{Z}_2^4 \times \mathbb{Z}_8$ | $1 \rightarrow \Pi_{13} \times \Pi_3 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | $2^2, 4^2$ | $2^2, 4^2$ | $G(16, 3)$ | 1 | $\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$ | $1 \rightarrow \Pi_5 \times \Pi_5 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | $2^3, 4$ | 2^6 | $D_4 \times \mathbb{Z}_2$ | 1 | $\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$ | $1 \rightarrow \Pi_9 \times \Pi_3 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | 2^5 | 2^5 | \mathbb{Z}_2^4 | 1 | \mathbb{Z}_2^4 | $1 \rightarrow \Pi_5 \times \Pi_5 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | 3^4 | 3^4 | \mathbb{Z}_3^2 | 1 | \mathbb{Z}_3^4 | $1 \rightarrow \Pi_4 \times \Pi_4 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 8 | \emptyset | 2^5 | 2^6 | \mathbb{Z}_2^3 | 1 | \mathbb{Z}_2^6 | $1 \rightarrow \Pi_3 \times \Pi_5 \rightarrow \pi_1 \rightarrow G \rightarrow 1$ |
| 6 | $1/2^2$ | $2^3, 4$ | $2^4, 4$ | $\mathbb{Z}_2 \times D_4$ | 1 | $\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$ | $1 \rightarrow \mathbb{Z}^2 \times \Pi_2 \rightarrow \pi_1 \rightarrow \mathbb{Z}_2^2 \rightarrow 1$ |
| 6 | $1/2^2$ | $2^4, 4$ | $2, 4, 6$ | $\mathbb{Z}_2 \times \mathfrak{S}_4$ | 1 | $\mathbb{Z}_2^3 \times \mathbb{Z}_4$ | $1 \rightarrow \Pi_2 \rightarrow \pi_1 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \rightarrow 1$ |
| 6 | $1/2^2$ | $2, 5^2$ | $2, 3^3$ | \mathfrak{A}_5 | 1 | $\mathbb{Z}_3 \times \mathbb{Z}_{15}$ | $\mathbb{Z}^2 \rtimes \mathbb{Z}_{15}$ |
| 6 | $1/2^2$ | $2, 4, 10$ | $2, 4, 6$ | $\mathbb{Z}_2 \times \mathfrak{S}_5$ | 1 | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\mathfrak{S}_3 \times D_{4,5,-1}$ |
| 6 | $1/2^2$ | $2, 7^2$ | $3^2, 4$ | $\text{PSL}(2, 7)$ | 2 | \mathbb{Z}_{21} | $\mathbb{Z}_7 \times \mathfrak{A}_4$ |
| 6 | $1/2^2$ | $2, 5^2$ | $3^2, 4$ | \mathfrak{A}_6 | 2 | \mathbb{Z}_{15} | $\mathbb{Z}_5 \times \mathfrak{A}_4$ |
| 5 | $1/3, 2/3$ | $2, 4, 6$ | $2^4, 3$ | $\mathbb{Z}_2 \times \mathfrak{S}_4$ | 1 | $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ | $1 \rightarrow \mathbb{Z}^2 \rightarrow \pi_1 \rightarrow D_{2,8,3} \rightarrow 1$ |
| 5 | $1/3, 2/3$ | $2^4, 3$ | $3, 4^2$ | \mathfrak{S}_4 | 1 | $\mathbb{Z}_2^2 \times \mathbb{Z}_8$ | $1 \rightarrow \mathbb{Z}^2 \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1$ |
| 5 | $1/3, 2/3$ | $4^2, 6$ | $2^3, 3$ | $\mathbb{Z}_2 \times \mathfrak{S}_4$ | 1 | $\mathbb{Z}_2^2 \times \mathbb{Z}_8$ | $1 \rightarrow \mathbb{Z}^2 \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1$ |
| 5 | $1/3, 2/3$ | $2, 5, 6$ | $3, 4^2$ | \mathfrak{S}_5 | 1 | \mathbb{Z}_8 | $D_{8,5,-1}$ |
| 5 | $1/3, 2/3$ | $3, 5^2$ | $2^3, 3$ | \mathfrak{A}_5 | 1 | $\mathbb{Z}_2 \times \mathbb{Z}_{10}$ | $\mathbb{Z}_5 \times Q_8$ |
| 5 | $1/3, 2/3$ | $2^3, 3$ | $3, 4^2$ | $\mathbb{Z}_2^4 \rtimes \mathfrak{S}_3$ | 1 | $\mathbb{Z}_2 \times \mathbb{Z}_8$ | $D_{8,4,3}?$ |
| 5 | $1/3, 2/3$ | $3, 5^2$ | $2^3, 3$ | \mathfrak{A}_5 | 1 | $\mathbb{Z}_2 \times \mathbb{Z}_{10}$ | $\mathbb{Z}_2 \times \mathbb{Z}_{10}$ |
| 4 | $1/2^4$ | 2^5 | 2^5 | \mathbb{Z}_2^3 | 1 | $\mathbb{Z}_2^3 \times \mathbb{Z}_4$ | $1 \rightarrow \mathbb{Z}^4 \rightarrow \pi_1 \rightarrow \mathbb{Z}_2^2 \rightarrow 1$ |
| 4 | $1/2^4$ | $2^2, 4^2$ | $2^2, 4^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | 1 | $\mathbb{Z}_2^3 \times \mathbb{Z}_4$ | $1 \rightarrow \mathbb{Z}^4 \rightarrow \pi_1 \rightarrow \mathbb{Z}_2^2 \rightarrow 1$ |
| 4 | $1/2^4$ | 2^5 | $2^3, 4$ | $\mathbb{Z}_2 \times D_4$ | 1 | $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ | $1 \rightarrow \mathbb{Z}^2 \rightarrow \pi_1 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \rightarrow 1$ |
| 4 | $1/2^4$ | $3, 6^2$ | $2^2, 3^2$ | $\mathbb{Z}_3 \times \mathfrak{S}_3$ | 1 | \mathbb{Z}_3^2 | $\mathbb{Z}^2 \rtimes \mathbb{Z}_3$ |
| 4 | $1/2^4$ | $3, 6^2$ | $2, 4, 5$ | \mathfrak{S}_5 | 1 | \mathbb{Z}_3^2 | $\mathbb{Z}^2 \rtimes \mathbb{Z}_3$ |
| 4 | $1/2^4$ | 2^5 | $2, 4, 6$ | $\mathbb{Z}_2 \times \mathfrak{S}_4$ | 1 | \mathbb{Z}_2^3 | $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$ |
| 4 | $1/2^4$ | $2^2, 4^2$ | $2, 4, 6$ | $\mathbb{Z}_2 \times \mathfrak{S}_4$ | 1 | $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ | $\mathbb{Z}^2 \rtimes \mathbb{Z}_4$ |
| 4 | $1/2^4$ | 2^5 | $3, 4^2$ | \mathfrak{S}_4 | 1 | $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ | $\mathbb{Z}^2 \rtimes \mathbb{Z}_4$ |
| 4 | $1/2^4$ | $2^3, 4$ | $2^3, 4$ | $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$ | 1 | \mathbb{Z}_4^2 | $G(32, 2)$ |
| 4 | $1/2^4$ | $2, 5^2$ | $2^2, 3^2$ | \mathfrak{A}_5 | 1 | \mathbb{Z}_{15} | \mathbb{Z}_{15} |
| 4 | $1/2^4$ | $2^2, 3^2$ | $2^2, 3^2$ | $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$ | 1 | \mathbb{Z}_3^3 | \mathbb{Z}_3^3 |
| 4 | $2/5^2$ | $2^3, 5$ | $3^2, 5$ | \mathfrak{A}_5 | 1 | $\mathbb{Z}_2 \times \mathbb{Z}_6$ | $\mathbb{Z}_2 \times \mathbb{Z}_6$ |
| 4 | $2/5^2$ | $2, 4, 5$ | $4^2, 5$ | $\mathbb{Z}_2^4 \rtimes D_5$ | 3 | \mathbb{Z}_8 | $\mathbb{Z}_8?$ |
| 4 | $2/5^2$ | $2, 4, 5$ | $3^2, 5$ | \mathfrak{A}_6 | 1 | \mathbb{Z}_6 | \mathbb{Z}_6 |

TABLE 2. Minimal product-quotient surfaces of general type with $p_g = 0$, $K^2 \leq 3$

| K_S^2 | Sing X | t_1 | t_2 | G | N | $H_1(S, \mathbb{Z})$ | $\pi_1(S)$ |
|---------|-------------------------------------|-------------|------------|--|---|------------------------------------|------------------------------------|
| 3 | 1/5, 4/5 | $2^3, 5$ | $3^2, 5$ | \mathfrak{A}_5 | 1 | $\mathbb{Z}_2 \times \mathbb{Z}_6$ | $\mathbb{Z}_2 \times \mathbb{Z}_6$ |
| 3 | 1/5, 4/5 | 2, 4, 5 | $4^2, 5$ | $\mathbb{Z}_2^4 \rtimes D_5$ | 3 | \mathbb{Z}_8 | $\mathbb{Z}_8?$ |
| 3 | 1/3, 1/2 ² , 2/3 | $2^2, 3, 4$ | 2, 4, 6 | $\mathbb{Z}_2 \times \mathfrak{S}_4$ | 1 | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ |
| 3 | 1/5, 4/5 | 2, 4, 5 | $3^2, 5$ | \mathfrak{A}_6 | 1 | \mathbb{Z}_6 | \mathbb{Z}_6 |
| 2 | 1/3 ² , 2/3 ² | $2, 6^2$ | $2^2, 3^2$ | $\mathbb{Z}_2 \times \mathfrak{A}_4$ | 1 | \mathbb{Z}_2^2 | Q_8 |
| 2 | 1/2 ⁶ | 4^3 | 4^3 | \mathbb{Z}_4^2 | 1 | \mathbb{Z}_2^3 | \mathbb{Z}_2^3 |
| 2 | 1/2 ⁶ | $2^3, 4$ | $2^3, 4$ | $\mathbb{Z}_2 \times D_4$ | 1 | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $\mathbb{Z}_2 \times \mathbb{Z}_4$ |
| 2 | 1/3 ² , 2/3 ² | $2^2, 3^2$ | $3, 4^2$ | \mathfrak{S}_4 | 1 | \mathbb{Z}_8 | \mathbb{Z}_8 |
| 2 | 1/3 ² , 2/3 ² | $3^2, 5$ | $3^2, 5$ | $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$ | 2 | \mathbb{Z}_5 | $\mathbb{Z}_5?$ |
| 2 | 1/2 ⁶ | $2, 5^2$ | $2^3, 3$ | \mathfrak{A}_5 | 1 | \mathbb{Z}_5 | \mathbb{Z}_5 |
| 2 | 1/2 ⁶ | $2^3, 4$ | 2, 4, 6 | $\mathbb{Z}_2 \times \mathfrak{S}_4$ | 1 | \mathbb{Z}_2^2 | \mathbb{Z}_2^2 |
| 2 | 1/3 ² , 2/3 ² | $3^2, 5$ | $2^3, 3$ | \mathfrak{A}_5 | 1 | \mathbb{Z}_2^2 | \mathbb{Z}_2^2 |
| 2 | 1/2 ⁶ | 2, 3, 7 | 4^3 | $\text{PSL}(2, 7)$ | 2 | \mathbb{Z}_2^2 | \mathbb{Z}_2^2 |
| 2 | 1/2 ⁶ | $2, 6^2$ | $2^3, 3$ | $\mathfrak{S}_3 \times \mathfrak{S}_3$ | 1 | \mathbb{Z}_3 | \mathbb{Z}_3 |
| 2 | 1/2 ⁶ | $2, 6^2$ | 2, 4, 5 | \mathfrak{S}_5 | 1 | \mathbb{Z}_3 | \mathbb{Z}_3 |
| 2 | 1/4, 1/2 ² , 3/4 | 2, 4, 7 | $3^2, 4$ | $\text{PSL}(2, 7)$ | 2 | \mathbb{Z}_3 | \mathbb{Z}_3 |
| 2 | 1/4, 1/2 ² , 3/4 | 2, 4, 5 | $3^2, 4$ | \mathfrak{A}_6 | 2 | \mathbb{Z}_3 | \mathbb{Z}_3 |
| 2 | 1/4, 1/2 ² , 3/4 | 2, 4, 5 | 3, 4, 6 | \mathfrak{S}_5 | 2 | \mathbb{Z}_3 | \mathbb{Z}_3 |
| 1 | 1/3, 1/2 ⁴ , 2/3 | $2^3, 3$ | $3, 4^2$ | \mathfrak{S}_4 | 1 | \mathbb{Z}_4 | \mathbb{Z}_4 |
| 1 | 1/3, 1/2 ⁴ , 2/3 | 2, 3, 7 | $3, 4^2$ | $\text{PSL}(2, 7)$ | 1 | \mathbb{Z}_2 | \mathbb{Z}_2 |
| 1 | 1/3, 1/2 ⁴ , 2/3 | 2, 4, 6 | $2^3, 3$ | $\mathbb{Z}_2 \times \mathfrak{S}_4$ | 1 | \mathbb{Z}_2 | \mathbb{Z}_2 |

Using Kimura's results ([Kim05], see also [GP03]), the present results, and those of the previous papers [BC04], [BCG08], [BCGP08], we get the following:

Corollary 0.6. *All the families in theorem 0.3 fulfill Bloch's conjecture, i.e., there are 77 families of surfaces of general type with $p_g = 0$ for which Bloch's conjecture holds.*

Let us briefly illustrate the strategy of proof for the above theorem and point out the difficulties arising in our more general situation.

Our goal is to find all product-quotient surfaces S of general type with $p_g = 0$.

Remark 0.7. 1) Let S be a surface of general type. Then $p_g(S) \geq q(S) := h^1(S, \mathcal{O}_S)$. In particular, $p_g = 0$ implies $q = 0$. If S is minimal, then $K_S^2 > 0$.

2) Let S be a product-quotient surface with quotient model $X = (C_1 \times C_2)/G$. If $q(S) = 0$, then $C_i/G \cong \mathbb{P}^1$. If S is of general type, then $g(C_i) \geq 2$.

By the above, we only need to recall the definition of a special case of an orbifold surface group: a polygonal group, (cf. [BCGP08] for the general situation).

Definition 0.8. A polygonal group of signature (m_1, \dots, m_r) is the group presented as follows:

$$\mathbb{T}(m_1, \dots, m_r) := \langle c_1, \dots, c_r | c_1^{m_1}, \dots, c_r^{m_r}, c_1 \cdot \dots \cdot c_r \rangle.$$

Let $p, p_1, \dots, p_r \in \mathbb{P}^1$ be $r + 1$ different points and for each $1 \leq i \leq r$ choose a simple geometric loop γ_i in $\pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_r\}, p)$ around p_i , such that $\gamma_1 \cdot \dots \cdot \gamma_r = 1$.

Then $\mathbb{T}(m_1, \dots, m_r)$ is the factor group of $\pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_r\}, p)$ by the subgroup normally generated by $\gamma_1^{m_1}, \dots, \gamma_r^{m_r}$.

Hence, by Riemann's existence theorem, any curve C together with an action of a finite group G on it such that $C/G \cong \mathbb{P}^1$ is determined (modulo automorphisms) by the following data:

- 1) the branch point set $\{p_1, \dots, p_r\} \subset \mathbb{P}^1$;
- 2) the kernel of the monodromy homomorphism $\pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_r\}, p) \rightarrow G$ which, once chosen loops γ_i as above, factors through $\mathbb{T}(m_1, \dots, m_r)$, where m_i is the branching index of p_i ; therefore giving the monodromy homomorphism is equivalent to give 2') an *appropriate orbifold* homomorphism

$$\varphi: \mathbb{T}(m_1, \dots, m_r) \rightarrow G,$$

i.e., a surjective homomorphism such that

- a) $\varphi(c_i)$ is an element of order exactly m_i and
- b) the *Hurwitz' formula* for the genus g of C holds:

$$2g - 2 = |G| \left(-2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right).$$

Therefore a product-quotient surface S of general type with $p_g = 0$ determines the following data

- a finite group G ;
- two sets of (branch) points in \mathbb{P}^1 ;
- two polygonal groups \mathbb{T}_1 and \mathbb{T}_2 ;
- (once chosen appropriate loops as above) two appropriate orbifold homomorphisms $\varphi_i: \mathbb{T}_i \rightarrow G$.

Vice versa, the data above determines the product-quotient surface.

The aim is to produce a Magma code which finds all possible $(G, \mathbb{T}_i, \varphi_i)$ yielding surfaces of general type with $p_g = 0$.

First of all we use the combinatorial restriction imposed by the assumption $p_g = 0$, and the condition that the quotient model of a product-quotient surface can only have cyclic quotient singularities.

This allows, for each value of $K^2 := K_S^2$, to restrict to a finite number of *baskets* of singularities (i.e., the combinatorial data given by the singular locus of X) and for each possible basket of singularities to a finite list of possible signatures t_1, t_2 of the respective polygonal groups.

Using prop. 1.13, a MAGMA ([BCP97]) script provides a finite list of possible signatures t_1, t_2 of the respective polygonal groups. The order of G is now determined by t_1, t_2 and by K^2 : it follows that there are only finitely many groups to consider.

A second MAGMA script computes, for each K^2 and each possible basket \mathcal{B} , all possible triples (t_1, t_2, G) , where G is a quotient of both polygonal groups (of respective signatures t_1, t_2) and has the right order. Note that our code skips a few pairs of signatures giving rise to groups of large order, either not covered by the MAGMA SmallGroup database, or causing extreme computational complexity. These cases left out by our program are then excluded via a case by case argument.

For each of the triples (t_1, t_2, G) in the output, there are several pairs of surjections (φ_1, φ_2) , each giving a product-quotient surface.

Recall that the triple (t_1, t_2, G) depends on a previously fixed basket \mathcal{B} . The product-quotient surface is a surface of general type with $p_g = 0$ and $K_S^2 = K^2$ if and only if the singularities are as prescribed.

A third MAGMA script produces the final list of surfaces, discarding the ones whose singular locus is not correct.

Observe that changing the choice of the loops γ_i (independently on both factors) and changing the G -action simultaneously on both Riemann surfaces by an automorphism of G , changes (φ_1, φ_2) , but does not change the isomorphism type of the resulting surface. Therefore the script returns only one representative for each equivalence class.

A last script calculates, using a result by Armstrong ([Arm65], [Arm68]), the fundamental groups.

In the case of infinite fundamental groups the structure theorem proven in [BCGP08] turns out to be extremely helpful to give an explicit description of these groups (since in general a presentation of a group does not say much about it).

Our code produces 73 families of product-quotient surfaces with $K_S^2 > 0$. While in the previous articles we have been done, here we do not know whether the minimal resolution of singularities S of the quotient model is in fact minimal. We have to develop methods in order to decide whether the product-quotient surface S is minimal, and in case it is not, to find the rational (-1) -curves on it. The construction of our surfaces is purely algebraic, the way from algebra to geometry is given by the Riemann existence theorem, which is not constructive. Therefore it is not straightforward how to get hold on delicate geometrical features of S .

We develop a criterion for the minimality of S arguing on the combinatorics of the basket of singularities. In all cases except one this criterion works and the minimality of S follows.

The remaining case turns out to be non minimal. We construct two very special singular G -invariant correspondences between C_1 and C_2 such that the respective strict transforms on S of their images in X are rational (-1) -curves. We prove that the surface obtained contracting these two (-1) -curves is minimal applying the previous criterion.

The paper is organized as follows:

In section 1 we discuss finite group actions on a product of compact Riemann surfaces of respective genera at least two, developping all the theory necessary to implement the algorithm.

In the second and third section we discuss the main classification algorithm.

Section 4 deals with rational curves of selfintersection (-1) on product-quotient surfaces. Here we give the criterion for the minimality of S , and show that it works for all the constructed surfaces except the fake Godeaux.

In section 5 we determine the minimal model of the fake Godeaux surface. The last section is devoted to comments about the computational complexity of the algorithms we used.

Finally, in a first appendix, we attach an expanded version of tables 1, 2 describing all the needed data if one wants to do explicit computations with one of the surfaces.

The second appendix is the MAGMA code we used.

1. THEORETICAL BACKGROUND

Let C_1, C_2 be two compact Riemann surfaces of respective genera $g_1, g_2 \geq 2$. Let G be a finite group acting faithfully on both curves and consider the diagonal action of G on $C_1 \times C_2$. This determines a product-quotient surface S , the minimal resolution of the singularities of $X := (C_1 \times C_2)/G$.

Remark 1.1. 1) Note that there are finitely many points on $C_1 \times C_2$ with non trivial stabilizer, which is automatically cyclic. Hence the quotient surface $X := (C_1 \times C_2)/G$ has a finite number of cyclic quotient singularities.

Recall that every cyclic quotient singularity is locally analytically isomorphic to the quotient of \mathbb{C}^2 by the action of a diagonal linear automorphism with eigenvalues $\exp(\frac{2\pi i}{n})$, $\exp(\frac{2\pi i a}{n})$ with $\text{g.c.d.}(a, n) = 1$; this is called a *singularity of type $\frac{1}{n}(1, a)$* .

2) We denote by K_X the canonical (Weil) divisor on the normal surface X corresponding to $i_*(\Omega_{X^0}^2)$, $i: X^0 \rightarrow X$ being the inclusion of the smooth locus of X . According to Mumford we have an intersection product with values in \mathbb{Q} for Weil divisors on a normal surface, and in particular we may consider the selfintersection

of the canonical divisor,

$$(1) \quad K_X^2 = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{|G|} \in \mathbb{Q},$$

which is not necessarily an integer.

3) It is well known that the exceptional divisor E of the minimal resolution of a cyclic quotient singularity of type $\frac{1}{n}(1, a)$ is a *Hirzebruch-Jung string*, i.e., $E = \bigcup_{i=1}^l E_i$ where all E_i are smooth rational curves, $E_i^2 = -b_i$, $E_i \cdot E_{i+1} = 1$ for $i \in \{1, \dots, l-1\}$ and $E_i \cdot E_j = 0$ otherwise. The b_i are given by the formula

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}}.$$

4) Since the minimal resolution $\pi: S \rightarrow X$ of the singularities of X replaces each singular point by a tree of smooth rational curves, we have, by van Kampen's theorem, that $\pi_1(X) = \pi_1(S)$.

5) Moreover, we have (in a neighbourhood of x)

$$K_S = \pi^* K_X + \sum_{i=1}^l a_i E_i,$$

where the rational numbers a_i are determined by the conditions

$$(K_S + E_j)E_j = -2, \quad (K_S - \sum_{i=1}^l a_i E_i)E_j = 0, \quad \forall j = 1, \dots, l.$$

The above formulae allow us to calculate the self intersection number of the canonical divisor K_S . In fact, we need the following

Definition 1.2. *Let X be a normal complex surface and suppose that the singularities of X are cyclic quotient singularities. Then we define the basket of singularities of X to be the multiset*

$$\mathcal{B}(X) := \left\{ \lambda \times \left(\frac{1}{n}(1, a) \right) : X \text{ has exactly } \lambda \text{ singularities of type } \frac{1}{n}(1, a) \right\}.$$

I.e., $\mathcal{B}(X) = \{2 \times \frac{1}{3}(1, 1), \frac{1}{4}(1, 3)\}$ means that the singular locus of X consists of two $\frac{1}{3}(1, 1)$ -points and one $\frac{1}{4}(1, 3)$ -point.

Remark 1.3. Note that in the definition of $\mathcal{B}(X)$ there is some ambiguity since singular points of type $\frac{1}{n}(1, a)$ are also of type $\frac{1}{n}(1, a')$ where $a' = a^{-1}$ in $(\mathbb{Z}/n\mathbb{Z})^*$. Therefore, e.g.,

$$\{2 \times \frac{1}{5}(1, 2)\} = \{1 \times \frac{1}{5}(1, 2), 1 \times \frac{1}{5}(1, 3)\} = \{2 \times \frac{1}{5}(1, 3)\}.$$

We consider these different representations as equal and usually do not distinguish between them.

Definition 1.4. *Let x be a singularity of type $\frac{1}{n}(1, a)$ with $\gcd(n, a) = 1$ and let $1 \leq a' \leq n - 1$ such that $a' = a^{-1}$ in $(\mathbb{Z}/n\mathbb{Z})^*$. Moreover, write $\frac{n}{a}$ as a continued fraction:*

$$\frac{n}{a} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \dots}} =: [b_1, \dots, b_l].$$

Then we define the following correction terms:

- i) $k_x := k(\frac{1}{n}(1, a)) := -2 + \frac{2+a+a'}{n} + \sum (b_i - 2) \geq 0$;
- ii) $e_x := e(\frac{1}{n}(1, a)) := l + 1 - \frac{1}{n} \geq 0$;
- iii) $B_x := 2e_x + k_x$.

Let \mathcal{B} be the basket of singularities of X (recall that X is normal and has only cyclic quotient singularities). Then we use the following notation

$$k(\mathcal{B}) := \sum_{x \in \mathcal{B}} k_x, \quad e(\mathcal{B}) := \sum_{x \in \mathcal{B}} e_x, \quad B(\mathcal{B}) := \sum_{x \in \mathcal{B}} B_x.$$

Proposition 1.5 ([BCGP08], prop. 2.6, and [MP10], cor. 3.6). *Let $S \rightarrow X := (C_1 \times C_2)/G$ be the minimal resolution of singularities of X . Then we have the following two formulae for the self intersection of the canonical divisor of S and the topological Euler characteristic of S :*

$$K_S^2 = \frac{8(g_1 - 1)(g_2 - 1)}{|G|} - k(\mathcal{B});$$

$$e(S) = \frac{4(g_1 - 1)(g_2 - 1)}{|G|} + e(\mathcal{B}).$$

A direct consequence of the above is the following:

Corollary 1.6. *Let $S \rightarrow X := (C_1 \times C_2)/G$ be the minimal resolution of singularities of X . Then*

$$K_S^2 = 8\chi(S) - \frac{1}{3}B(\mathcal{B}).$$

Proof. By prop. 1.5 we have

$$e(S) = \frac{K_S^2 + B(\mathcal{B})}{2}.$$

By Noether's formula we obtain

$$12\chi(S) = K_S^2 + e(S) = \frac{3K_S^2 + B(\mathcal{B})}{2}$$

□

We shall now list some properties of the basket of singularities of the quotient model $X = (C_1 \times C_2)/G$ of a product-quotient surface.

Lemma 1.7. *Let $X = (C_1 \times C_2)/G$ be as above. There exists a representation of the basket (cf. remark 1.3)*

$$\mathcal{B}(X) = \left\{ \lambda_1 \times \frac{1}{n_1}(1, a_1), \dots, \lambda_R \times \frac{1}{n_R}(1, a_R) \right\}$$

such that

$$\sum \lambda_i \frac{a_i}{n_i} \in \mathbb{Z}.$$

Proof. Consider the fibration $X \rightarrow C_1/G$, and let F_1, \dots, F_r be the singular fibres taken with the reduced structure. Let \tilde{F}_i be the strict transform of F_i on S .

Then, by [P10, Proposition 2.8], for a suitable representation of the basket

$$\sum \lambda_i \frac{a_i}{n_i} = - \sum \tilde{F}_i^2 \in \mathbb{Z}.$$

□

Definition 1.8. *A multiset*

$$\mathcal{B} := \left\{ \lambda_1 \times \frac{1}{n_1}(1, a_1), \dots, \lambda_R \times \frac{1}{n_R}(1, a_R) \right\}$$

is called a possible basket of singularities for (K^2, χ) if and only if it satisfies the following conditions:

- there is a representation of \mathcal{B} , say

$$\mathcal{B} := \left\{ \lambda'_1 \times \frac{1}{n'_1}(1, a'_1), \dots, \lambda'_{R'} \times \frac{1}{n'_{R'}}(1, a'_{R'}) \right\}$$

such that $\sum \lambda'_i \frac{a'_i}{n'_i} \in \mathbb{Z}$,

- $B(\mathcal{B}) = 3(8\chi(S) - K^2)$.

It is now obvious that the basket of the quotient model X of a product-quotient surface S is a possible basket of singularities for $(K_S^2, \chi(\mathcal{O}_S))$.

1.1. Finiteness of the classification problem. The next lemma shows that, for every pair $(K^2, \chi) \in \mathbb{Z} \times \mathbb{Z}$, there are only finitely many possible baskets of singularities for (K^2, χ) .

Lemma 1.9. *Let $C \in \mathbb{Q}$ be fixed. Then there are finitely many baskets \mathcal{B} such that*

$$B(\mathcal{B}) = C.$$

More precisely, we have:

- i) $|\mathcal{B}| \leq \frac{C}{3}$,
- ii) if $\lambda \times \frac{1}{n}(1, a) \in \mathcal{B}$ and $\frac{n}{a} = [b_1, \dots, b_l]$, then $\lambda \sum b_i \leq C$.

Proof. Observe first that $B(\frac{1}{n}(1, a)) = \frac{a+a'}{n} + \sum b_i \geq 3$. In particular,

$$C = B(\mathcal{B}) \geq 3|\mathcal{B}|,$$

which shows (i). (ii) is obvious. □

Remark 1.10. Note that, by [Ser96], if $S \rightarrow X = C_1 \times C_2/G$ is a product-quotient surface, then $q(S) = g(C_1/G) + g(C_2/G)$. Therefore $q(S) = 0 \Leftrightarrow g(C_1/G) = g(C_2/G) = 0$. This implies that a product-quotient surface S of general type with quotient model $X = (C_1 \times C_2)/G$ has $p_g(S) = 0$ if and only if

- $\chi(\mathcal{O}_S) = 1$ and
- $C_1/G \cong C_2/G \cong \mathbb{P}^1$.

From now on we shall restrict ourselves to product-quotient surfaces S of general type with $p_g(S) = 0$. Let $\lambda_i: C_i \rightarrow \mathbb{P}^1$, $i = 1, 2$ be the two Galois covers associated to it.

Recall that, by Riemann's existence theorem (cf. the introduction for more details), an action of a finite group G on a compact Riemann surface C of genus g such that $C/G \cong \mathbb{P}^1$ is given by an appropriate orbifold homomorphism

$$\varphi: \mathbb{T}(m_1, \dots, m_r) \rightarrow G$$

such that the Riemann-Hurwitz relation holds:

$$2g - 2 = |G| \left(-2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) \right).$$

Then $\lambda_i: C_i \rightarrow \mathbb{P}^1$, $i = 1, 2$ induce two appropriate orbifold homomorphisms

$$\varphi_1: \mathbb{T}(m_1, \dots, m_r) \rightarrow G,$$

$$\varphi_2: \mathbb{T}(n_1, \dots, n_s) \rightarrow G.$$

Here λ_1 is branched in r points $p_1, \dots, p_r \in \mathbb{P}^1$ with branching indices m_1, \dots, m_r , and λ_2 is branched in s points $p'_1, \dots, p'_s \in \mathbb{P}^1$ with branching indices n_1, \dots, n_s .

We need the following

Definition 1.11. Fix an r -tuple of natural numbers $t := (m_1, \dots, m_r)$ and a basket of singularities \mathcal{B} . Then we associate to these the following numbers:

$$\Theta(t) := -2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right);$$

$$\alpha(t, \mathcal{B}) := \frac{12 + k(\mathcal{B}) - e(\mathcal{B})}{6\Theta(t)}.$$

Moreover, we recall the following

Definition 1.12. *The minimal positive integer I_x such that $I_x K_X$ is Cartier in x is called the index of the singularity x .*

The index of X is the minimal positive integer I such that IK_X is Cartier. In particular, $I = \text{lcm}_{x \in \text{Sing } X} I_x$.

It is well known (cf. e.g. [Mat02], theorem 4-6-20) that the index of a cyclic quotient singularity $\frac{1}{n}(1, a)$ is

$$I_x = \frac{n}{\gcd(n, a+1)}.$$

By lemma 1.9, fixed $K^2 \in \mathbb{Z}$, there are finitely many possible baskets of singularities for $(K^2, \chi(\mathcal{O}_S) = 1)$.

We shall bound now, for fixed K^2 and \mathcal{B} , the possibilities for:

- $|G|$,
- $t_1 := (m_1, \dots, m_r)$,
- $t_2 := (n_1, \dots, n_s)$,

of a product-quotient surface S with $K_S^2 = K^2$ and basket of singularities of the quotient model X equal to \mathcal{B} .

Proposition 1.13. *Fix $K^2 \in \mathbb{Z}$, and fix a possible basket of singularities \mathcal{B} for $(K^2, 1)$. Let S be a product-quotient surface S of general type such that*

- i) $p_g(S) = 0$,
- ii) $K_S^2 = K^2$,
- iii) *the basket of singularities of the quotient model $X = (C_1 \times C_2)/G$ of S equals \mathcal{B} .*

Then:

- a) $g(C_1) = \alpha(t_2, \mathcal{B}) + 1$, $g(C_2) = \alpha(t_1, \mathcal{B}) + 1$;
- b) $|G| = \frac{8\alpha(t_1, \mathcal{B})\alpha(t_2, \mathcal{B})}{K^2 + k(\mathcal{B})}$;
- c) $r, s \leq \frac{K^2 + k(\mathcal{B})}{2} + 4$;
- d) m_i divides $2\alpha(t_1, \mathcal{B})I$, n_j divides $2\alpha(t_2, \mathcal{B})I$;
- e) *there are at most $|\mathcal{B}|/2$ indices i such that m_i does not divide $\alpha(t_1, \mathcal{B})$, and similarly for the n_j ;*
- f) $m_i \leq \frac{1+I \frac{K^2 + k(\mathcal{B})}{2}}{f(t_1)}$, $n_i \leq \frac{1+I \frac{K^2 + k(\mathcal{B})}{2}}{f(t_2)}$, where I is the index of X , and $f(t_1) := \max(\frac{1}{6}, \frac{r-3}{2})$, $f(t_2) := \max(\frac{1}{6}, \frac{s-3}{2})$;

g) *except for at most $|\mathcal{B}|/2$ indices i , the sharper inequality $m_i \leq \frac{1 + \frac{K^2 + k(\mathcal{B})}{4}}{f(t_1)}$ holds, and similarly for the n_j .*

Remark 1.14. Note that prop. 1.13, b) shows that t_1, t_2 determine the order of G . c), f) imply that there are only finitely many possibilities for the types t_1, t_2 . Parts d), e) and g) are strictly necessary to obtain an efficient algorithm.

Proof. a) Observe that by corollary 1.6, since $\chi(\mathcal{O}_S) = 1$, we have

$$\Theta(t_1)\alpha(t_1, \mathcal{B}) = \frac{12 + k(\mathcal{B}) - e(\mathcal{B})}{6} = \frac{24 - B(\mathcal{B}) + 3k(\mathcal{B})}{12} = \frac{K^2 + k(\mathcal{B})}{4}$$

and then by prop. 1.5 and Hurwitz' formula

$$\alpha(t_1, \mathcal{B}) = \frac{K^2 + k(\mathcal{B})}{4\Theta(t_1)} = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{4|G|(-2 + \sum_{i=1}^r(1 - \frac{1}{m_i}))} = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{4(2g(C_1) - 2)}.$$

b)

$$|G| = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{K^2 + k(\mathcal{B})} = \frac{8\alpha(t_2, \mathcal{B})\alpha(t_1, \mathcal{B})}{K^2 + k(\mathcal{B})}.$$

c) Note that $r \leq 2 \sum_{i=1}^r(1 - \frac{1}{m_i}) = 2\Theta(t_1) + 4$. On the other hand, since $g(C_j) \geq 2$, we have $1 \leq \alpha(t_i, \mathcal{B}) = \frac{K^2 + k(\mathcal{B})}{4\Theta(t_i)}$. This implies that $(0 <) \Theta(t_i) \leq \frac{K^2 + k(\mathcal{B})}{4}$.

d) Each m_i is the branching index of a branch point p_i of $\lambda_1: C_1 \rightarrow C_1/G \cong \mathbb{P}^1$. Let F_i be the fibre over p_i of the map $X \rightarrow C_1/G$. Then $F_i = m_i W_i$ for some irreducible Weil divisor W_i .

$$2\alpha(t_1, \mathcal{B}) = 2g(C_2) - 2 = K_X F_i = m_i K_X W_i.$$

Therefore

$$\frac{2\alpha(t_1, \mathcal{B})I}{m_i} = (IK_X)W_i \in \mathbb{Z}.$$

e) By [Ser96], if F_i contains a singular point of X , then it contains at least 2 singular points. Therefore there are at most $|\mathcal{B}|/2$ indices i ($1 \leq i \leq r$) such that $F_i \cap \text{Sing } X \neq \emptyset$.

For all other indices j we have $F_j \cap \text{Sing } X = \emptyset$. Then W_j is Cartier and K_X is Cartier in a neighbourhood of W_j . In particular, $\frac{\alpha(t_1, \mathcal{B})I}{m_j} = \frac{K_X W_j}{2} \in \mathbb{Z}$.

f) Note that $\Theta(t_1) + \frac{1}{m_i} \geq \frac{r-3}{2}$. Moreover, $\Theta(t_1) > 0$ implies that $r \geq 3$. Obviously, if $r = 3$, since $\Theta(2, 2, m) = -\frac{1}{m} < 0$, then $\Theta(t_1) + \frac{1}{m_i} \geq \frac{1}{6}$. Therefore $\Theta(t_1) + \frac{1}{m_i} \geq f(t_1)$, whence $m_i \leq \frac{1 + \Theta(t_1)m_i}{f(t_1)}$.

By d) $m_i \leq 2\alpha(t_1, \mathcal{B})I = \frac{K^2+k(\mathcal{B})}{2\Theta(t_1)}I$. This implies

$$m_i \leq \frac{1 + \Theta(t_1)m_i}{f(t_1)} \leq \frac{1 + \Theta(t_1)\frac{K^2+k(\mathcal{B})}{2\Theta(t_1)}I}{f(t_1)} = \frac{1 + \frac{K^2+k(\mathcal{B})}{2}I}{f(t_1)}.$$

g) This is proved by the same argument as in f), using e) instead of d). \square

1.2. How to read the basket \mathcal{B} from the group theoretical data. Our next goal is to describe explicitly how the two appropriate orbifold homomorphisms

$$\varphi_1: \mathbb{T}(m_1, \dots, m_r) \rightarrow G,$$

$$\varphi_2: \mathbb{T}(n_1, \dots, n_s) \rightarrow G.$$

determine the singularities of the quotient model X .

We denote the images of the standard generators (the c_i in definition 0.8) of $\mathbb{T}(m_1, \dots, m_r)$ (resp. of $\mathbb{T}(n_1, \dots, n_s)$) by (g_1, \dots, g_r) (resp. by (h_1, \dots, h_s)).

Moreover we set $H_i := \langle g_i \rangle$ and $H'_j := \langle h_j \rangle$.

We have now the following commutative diagram:

$$(2) \quad \begin{array}{ccccc} & & C_1 \times C_2 & & \\ & \swarrow p_1 & \downarrow \lambda_{12} & \searrow p_2 & \\ C_1 & & X = (C_1 \times C_2)/G & & C_2 \\ \downarrow \lambda_1 & \swarrow f_1 & \downarrow \lambda & \searrow f_2 & \downarrow \lambda_2 \\ C_1/G \cong \mathbb{P}^1 & & C_1/G \times C_2/G \cong \mathbb{P}^1 \times \mathbb{P}^1 & & C_2/G \cong \mathbb{P}^1 \end{array}$$

Note that the singular points of X are the points $Q = \lambda_{12}(q, q')$ such that the stabilizer

$$\text{Stab}(q, q') := \text{Stab}(q) \cap \text{Stab}(q') \neq \{1\}.$$

In particular, if $Q \in \text{Sing}(X)$ then $\lambda(Q) = (p_i, p'_j)$, where p_i (resp. p'_j) is a critical value of λ_1 (resp. λ_2).

We first prove the following

Proposition 1.15. *Let $i \in \{1, \dots, r\}$, $j \in \{1, \dots, s\}$. Then*

- (1) *there is a G -equivariant bijective map $(\lambda \circ \lambda_{12})^{-1}(p_i, p'_j) \rightarrow G/H_i \times G/H'_j$, where the G -action on the target is given by left multiplication (simultaneously on both factors);*
- (2) *intersecting with $\{\bar{1}\} \times G/H'_j$ gives a bijection between the orbits of the above G -action on $G/H_i \times G/H'_j$ with the orbits of the H_i -action on G/H'_j , i.e. with $(G/H'_j)/H_i$*

Proof. 1) Wlog we can assume $(i, j) = (1, 1)$. We fix the following notation

$$\pi_1^{-1}(p_1) = \{q_1, \dots, q_k\}, \quad \pi_2^{-1}(p'_1) = \{q'_1, \dots, q'_l\}.$$

There is a G -equivariant bijection between $\{q_1, \dots, q_k\}$ and the set of left cosets

$$\{a_1 H_1, \dots, a_k H_1\},$$

mapping each q_j in $\{g \in G | gq_1 = q_j\}$; similarly there is a bijection between $\{q'_1, \dots, q'_l\}$ and

$$\{a'_1 H'_1, \dots, a'_l H'_1\}.$$

This gives a G -equivariant bijection between $(\lambda \circ \lambda_{12})^{-1}(p_1, p'_1)$ and $G/H_1 \times G/H'_1$.

2) We consider the (diagonal) G -action on $G/H_1 \times G/H'_1$ by left multiplication. Note that the G -orbits are in one-to-one correspondence with the points of $\lambda((\lambda \circ \lambda_{12})^{-1}(p_1, p'_1))$.

Observe that

- i) $(hH_1, h'H'_1)$ is in the same G -orbit as $(H_1, h^{-1}h'H'_1)$;
- ii) (H_1, gH'_1) is in the same G -orbit as $(H_1, g'H'_1)$ if and only if gH'_1 and $g'H'_1$ are in the same orbit for the action of H_1 .

□

Remark 1.16. Recall that $\text{Sing}(X) \subset \lambda^{-1}(\{(p_i, p'_j)\})$. Observe moreover that proposition 1.15 gives for each (i, j) a bijection between $\lambda^{-1}(p_i, p'_j)$ and $(G/H'_j)/H_i$.

We still have to determine the types of the singularities. This is done in the following

Proposition 1.17. *An element $[g] \in (G/H'_j)/H_i$ corresponds to a point $\frac{1}{n}(1, a)$, where $n = |H_i \cap gH'_j g^{-1}|$, and a is given as follows: let δ_i be the minimal positive number such that there exists $1 \leq \gamma_j \leq o(h_j)$ with $g_i^{\delta_i} = gh_j^{\gamma_j} g^{-1}$. Then $a = \frac{n\gamma_j}{o(h_j)}$.*

Proof. Again we can assume wlog. that $(i, j) = (1, 1)$. Then $[g]$ corresponds to a (singular) point of type $\frac{1}{n}(1, a)$ with $n = |\text{Stab}(q_1, gq'_1)| = |H_1 \cap gH'_1 g^{-1}|$. Recall that $H_1 = \langle g_1 \rangle$, and $H'_1 = \langle h_1 \rangle$.

Let δ be the minimal positive number such that there is $\gamma \in \mathbb{N}$ (which can be chosen such that $1 \leq \gamma \leq o(h_1)$) such that $g_1^\delta = gh_1^\gamma g^{-1}$. Then $\langle g_1^\delta \rangle = \text{Stab}(q_1, gq'_1)$.

Therefore $o(g_1) = n\delta$. In local analytic coordinates (x, y) of $C_1 \times C_2$, g_1^δ acts as

$$e^{\frac{2\pi i}{n}} = e^{\frac{2\pi i \delta}{o(g_1)}}$$

on the variable x and as

$$e^{\frac{2\pi i a}{n}} = e^{\frac{2\pi i \gamma}{o(h_1)}}.$$

on the variable y . This shows that $a = \frac{n\gamma}{o(h_1)}$. \square

2. DESCRIPTION AND IMPLEMENTATION OF THE CLASSIFICATION ALGORITHM

Now we use the results of the previous section to write a MAGMA script to find all minimal surfaces S of general type with $p_g = 0$, which are product-quotient surfaces.

The full code is rather long and we attach a commented version in the appendix. We describe here the strategy, and explain the most important scripts.

First of all, by rem. 0.7, cor. 1.6, $1 \leq K_S^2 \leq 8$. The case $K_S^2 = 8$ has been classified in [BCG08].

Therefore we fix a value of $K^2 \in \{1, \dots, 7\}$.

Step 1: The script **Baskets** lists all the *possible baskets of singularities* for $(K^2, 1)$ as in definition 1.8. Indeed, there are only finitely many of them by lemma 1.9. The input is $3(8 - K^2)$, as in lemma 1.9, so to get *e.g.*, all baskets for $K_S^2 = 5$, we need to ask *Baskets(9)*.

Step 2: By proposition 1.13, once we know the basket of singularities of X , then there are finitely many possible signatures. **ListOfTypes** computes them using the inequalities we have proved in proposition 1.13. Here the input is K^2 , so **ListOfTypes** first computes *Baskets(3(8 - K²))* and then computes for each basket all numerically compatible signatures. The output is a list of pairs, the first element of each pair being a basket and the second element being the list of all signatures compatible with that basket.

Step 3: Every surface produces two signatures, one for each curve C_i , both compatible with the basket of singularities of X ; if we know the signatures and the basket, Proposition 1.13, b) tells us the order of G . **ListGroups**, whose input is K^2 , first computes *ListOfTypes(K²)*. Then for each pair of signatures in the output, it calculates the order of the group. Next it searches for the groups of the given order which admit appropriate orbifold homomorphism from the polygonal groups corresponding to both signatures. For each affirmative answer it stores the triple (basket, pair of signatures, group) in a list which is the main output.

The script has some shortcuts.

- If one of the signatures is $(2,3,7)$, then G , being a quotient of $\mathbb{T}(2,3,7)$, is perfect. MAGMA knows all perfect groups of order ≤ 50000 , and then **ListGroups** checks first if there are perfect group of the right order: if not, this case can't occur.

- If:
 - either the expected order of the group is 1024 or bigger than 2000, since MAGMA does not have a list of the finite groups of this order;
 - or the order is a number as *e.g.*, 1728, where there are too many isomorphism classes of groups;
 then ListGroups just stores these cases in a list, secondary output of the script. We will consider these "exceptional" cases in the next subsection, showing that they do not occur.

Step 4: ExistingSurfaces runs on the output of $ListGroups(K^2)$ and throws away all triples giving rise only to surfaces whose singularities do not correspond to the basket.

Step 5: Each triple in the output of $ExistingSurfaces(K^2)$ gives many different pairs of appropriate orbifold homomorphisms. In [BC04] (def. 1.2., thm. 1.3.) there is explicitly described an equivalence relation on such pairs of appropriate orbifold homomorphisms. By [BC04] thm. 1.3. and its proof if two pairs belong to the same equivalence class then the surfaces obtained by them (as described in the introduction, choosing for both the same points and the same loops) are isomorphic. More precisely, they are product-quotient surfaces with the same group G and the isomorphism is induced by a G -equivariant isomorphism of the related products of curves.

The script **FindSurfaces** produces, given a triple (basket, pair of types, group), only one representative for each equivalence class.

Step 6: Pi1 uses Armstrong's result ([Arm65], [Arm68]) to compute the fundamental group of each of the constructed surfaces.

Remark 2.1. We performed step 5 to avoid useless repetitions (note that the cardinality of some equivalence class is a few millions). Nevertheless, it is still possible that two different outputs of FindSurfaces give isomorphic surfaces. One of the reasons for running step 6 is indeed to show that this is in many cases not true, since the fundamental group distinguishes them even topologically.

We would also like to point out that, even if our families have a natural number of parameters, we do not make any claim on the dimension of the induced subsets of the Gieseker moduli space of the surfaces of general type.

Remark 2.2. The output of *Pi1* is a (sometimes rather complicated) presentation of the fundamental groups of the respective surfaces. We use the structure theorem on the fundamental group of product-quotient surfaces [BCGP08, Theorem 4.1] to give the (nicer) description of the fundamental groups in tables 1 and 2.

We have run *FindSurfaces* on each triple of the output of *ListGroups*(K^2), $K^2 \in \{1 \dots 7\}$. This has given all the families in tables 1 and 2, and one more, the “fake Godeaux surface”.

To prove theorem 0.3 it remains to show that

- all cases skipped by *ListGroups* do not occur,
- all the families in tables 1 and 2 are minimal surfaces of general type,
- the “fake Godeaux surface” has the properties in thm. 0.3, 3).

This will be accomplished in sections 3, 4 and 5.

product-quotient surfaces with $p_g = q = 0$ and $K_S^2 \geq 1$, as soon as we prove that the cases skipped by *ListGroups* cannot occur. This is done in the next section.

3. THE EXCEPTIONAL CASES

The cases skipped by *ListGroups* and stored in its secondary output are listed in table 3.

TABLE 3. Secondary output of *ListGroups*

| K^2 | Basket | t_1 | t_2 | $ G $ |
|-------|-------------------|---------|---------|-------|
| 6 | $1/2^2$ | 2, 3, 7 | 2, 4, 5 | 2520 |
| 5 | $2/3, 1/3$ | 2, 3, 8 | 2, 4, 6 | 768 |
| 5 | $2/3, 1/3$ | 2, 3, 8 | 2, 3, 7 | 2688 |
| 5 | $2/3, 1/3$ | 2, 3, 8 | 2, 3, 8 | 1536 |
| 5 | $2/3, 1/3$ | 2, 3, 8 | 2, 3, 9 | 1152 |
| 4 | $1/2, 1/4^2$ | 2, 3, 7 | 2, 4, 5 | 2520 |
| 4 | $1/2^4$ | 2, 3, 8 | 2, 3, 8 | 1152 |
| 2 | $1/4^4$ | 2, 4, 5 | 2, 3, 7 | 2520 |
| 2 | $1/2^3, 1/4^2$ | 2, 3, 8 | 2, 3, 8 | 1152 |
| 2 | $2/3^2, 1/3^2$ | 2, 3, 8 | 2, 3, 8 | 768 |
| 1 | $1/4, 1/5, 11/20$ | 2, 3, 8 | 2, 3, 8 | 2016 |
| 1 | $2/7^2, 1/7$ | 2, 3, 7 | 2, 3, 7 | 6048 |
| 1 | $1/4, 2/5, 3/20$ | 2, 3, 8 | 2, 3, 8 | 2016 |
| 1 | $1/4, 5/8, 1/8$ | 2, 3, 8 | 2, 3, 8 | 2016 |

In this section we shall show that all these cases do not occur. One of the main tools here is the script **ExSphGens**, which checks, given a finite group G and a signature, the existence of an appropriate orbifold homomorphism from the polygonal group of given signature to G .

Proposition 3.1. *There is no finite quotient of $\mathbb{T}(2, 3, 7)$ of order 2520, 2688 or 6048.*

Proof. A finite quotient of $\mathbb{T}(2, 3, 7)$ is perfect. The only perfect groups of order 2520 resp. 6048 are \mathfrak{A}_7 resp. $SU(3, 3)$; running the MAGMA script *ExSphGens* on these two groups, it turns out that both cannot be a quotient of $\mathbb{T}(2, 3, 7)$.

There are 3 perfect groups of order 2688. Let G be one of these three groups. Investigating their normal subgroups we find that G is either an extension of the form

$$1 \rightarrow (\mathbb{Z}/2\mathbb{Z})^3 \rightarrow G \rightarrow SU(2, 7) \rightarrow 1$$

or of the form

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow G \rightarrow \text{SmallGroup}(1344, 11686) \rightarrow 1.$$

Running *ExSphGens* on $SU(2, 7)$ and on $\text{SmallGroup}(1344, 11686)$, we see that none of them is quotient of $\mathbb{T}(2, 3, 7)$. Since $\mathbb{T}(3, 7) = \{1\}$, this implies that G is not a quotient of $\mathbb{T}(2, 3, 7)$. \square

Proposition 3.2. *There is no finite quotient of $\mathbb{T}(2, 3, 8)$ of order 1152 or 2016.*

Proof. Assume that G is a group of order 1152 or 2016 admitting a surjective homomorphism $\mathbb{T}(2, 3, 8) \rightarrow G$.

Since $\mathbb{T}(2, 3, 8)^{ab} \cong \mathbb{Z}/2\mathbb{Z}$, the abelianization of G is a quotient of $\mathbb{Z}/2\mathbb{Z}$ and since there are no perfect groups of order 1152 or 2016, $G^{ab} \cong \mathbb{Z}/2\mathbb{Z}$.

$|G| = 1152$. The following MAGMA computation

```
> for G in SmallGroups(1152) do
for> if #AbelianQuotient(G) eq 2 then
for|if> if ExSphGens(G,{2,3,8}) then
for|if|if> print G;
for|if|if> end if; end if; end for;
Warning: May return more than 100,000 groups -- this will take a
VERY long time. Would a SmallGroupProcess be more appropriate?
>
```

shows, that G can't have order 1152.

$|G| = 2016$. Since $G^{ab} \cong \mathbb{T}(2, 3, 8)^{ab}$, $[\mathbb{T}(2, 3, 8), \mathbb{T}(2, 3, 8)] \cong \mathbb{T}(3, 3, 4)$ surjects onto $[G, G]$ and therefore $[G, G]$ is a group of order 1008 admitting an appropriate orbifold homomorphism from $\mathbb{T}(3, 3, 4)$.

Since $\mathbb{T}(3, 3, 4)^{ab} \cong \mathbb{Z}/3\mathbb{Z}$ and since there are no perfect groups of order 1008, we get a contradiction running the following script.

```
> for G in SmallGroups(1008) do
for> if #AbelianQuotient(G) eq 3 then
for|if> if ExSphGens(G,{3,3,4}) then
for|if|if> print G;
```

```
for|if|if> end if; end if; end for;
>
```

□

Proposition 3.3. *There is exactly one group G of order 1536 admitting an appropriate orbifold homomorphism $\mathbb{T}(2, 3, 8) \rightarrow G$.*

There is no product-quotient surface of general type with $p_g = 0$ with group G , whose quotient model has $\{\frac{1}{3}(1, 1), \frac{1}{3}(1, 2)\}$ as basket of singularities.

Proof. There are 408641062 groups of order 1536. We have to use a "SmallGroup-Process" to deal with this case.

The claim follows, running the subsequent MAGMA script

```
> P:=SmallGroupProcess(1536);
> i:=1;
> repeat
repeat> G:=Current(P);
repeat> if #AbelianQuotient(G) eq 2 then
repeat|if> if ExSphGens(G,{2,3,8}) then
repeat|if|if> print i;
repeat|if|if> end if;
repeat|if> end if;
repeat> i:=i+1;
repeat> Advance(~P);
repeat> until IsEmpty(P);
408544637
> G:=SmallGroup(1536,408544637);
> FindSurfaces({*1/3,2/3*},{*{2,3,8}^2*},G);
{@ @}
>
```

□

Proposition 3.4. *1) There is exactly one group G of order 768 admitting an appropriate orbifold homomorphism $\mathbb{T}(2, 3, 8) \rightarrow G$.*

2) G does not admit an appropriate orbifold homomorphism $\mathbb{T}(2, 4, 6) \rightarrow G$.

3) There is no product-quotient surface of general type with $p_g = 0$ with group G , whose quotient model has $\{2 \times \frac{1}{3}(1, 1), 2 \times \frac{1}{3}(1, 2)\}$ as basket of singularities.

Proof. Using the same arguments as in the previous case, the two assertions follow from the following MAGMA computation:

```
> P:=SmallGroupProcess(768);
```

```

> repeat
repeat>  G:=Current(P);
repeat>  if #AbelianQuotient(G) eq 2 then
repeat|if>    if ExSphGens(G,{2,3,8}) then
repeat|if|if>      print IdentifyGroup(G);
repeat|if|if>    end if;
repeat|if>  end if;
repeat>  Advance(~P);
repeat> until IsEmpty(P);
<768, 1085341>
> G:=SmallGroup(768,1085341);
> ExSphGens(G,{2,4,6});
false
> FindSurfaces({*1/3^2,2/3^2*},{* {2,3,8}^2 *},G);
{@ @}
>

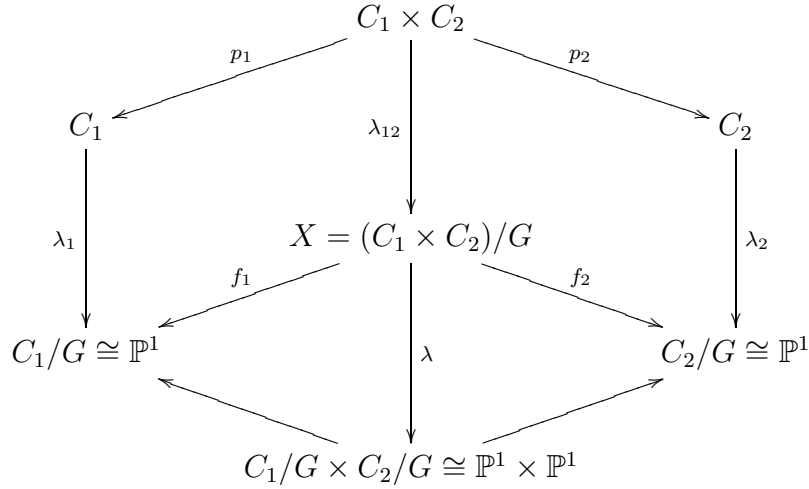
```

□

Propositions 3.1, 3.2, 3.3 and 3.4 exclude all cases in table 3.

4. RATIONAL CURVES ON PRODUCT-QUOTIENT SURFACES

We need to recall diagram 2:



Assume that $\Gamma \subset X$ is a (possibly singular) rational curve. Let $\bar{\Gamma} := \lambda_{12}^*(\Gamma) = \sum_1^k n_i \Gamma_i$ be the decomposition in irreducible components of its pull back to $C_1 \times C_2$.

Observe that $n_i = 1$, $\forall i$ (since λ_{12} has discrete ramification), and that G acts transitively on the set $\{\Gamma_i | i \in \{1, \dots, k\}\}$. Hence there is a subgroup $H \leq G$ of index k acting on Γ_1 such that $\lambda_{12}(\Gamma_1) = \Gamma_1/H = \Gamma$.

Normalizing Γ_1 and Γ , we get the following commutative diagram:

$$(3) \quad \begin{array}{ccc} \tilde{\Gamma}_1 & \longrightarrow & \Gamma_1 \\ \gamma \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\nu} & \Gamma \end{array}$$

and, since each automorphism lifts to the normalization, H acts on $\tilde{\Gamma}_1$ and γ is the quotient map $\tilde{\Gamma}_1 \rightarrow \tilde{\Gamma}_1/H \cong \mathbb{P}^1$.

Lemma 4.1. *Let p be a branch point of γ of multiplicity m . Then $\nu(p)$ is a singular point of X of type $\frac{1}{n}(1, a)$, where $m|n$.*

Proof. Let $p' \in \tilde{\Gamma}_1$ be a ramification point of γ and $g \in H$ a generator of its stabilizer. The stabilizer A of the image of p' in $C_1 \times C_2$ (with respect of the action of G) contains g , whence $m = o(g)$ divides $n = |A|$. \square

Remark 4.2. It follows from the Enriques-Kodaira classification of complex algebraic surfaces that, if $q(S) = 0$, either

- i) S is rational, or
- ii) S is of general type, or
- iii) $K_S^2 \leq 0$.

Remark 4.3. On a smooth surface S of general type every irreducible curve C with $K_S C \leq 0$ is smooth and rational.

Proof. Consider the morphism $f: S \rightarrow M$ to its minimal model. Assume that there is an irreducible curve $C \subset S$ with $K_S C \leq 0$ which is either singular or irrational. Then C is not contracted by f and $C' := f(C)$ is a still singular resp. irrational curve with $K_M C' \leq K_S C \leq 0$ which, by a classical argument (e.g. cf. [Bom73], prop. 1), implies that C' is a smooth rational curve of selfintersection (-2) , a contradiction. \square

Proposition 4.4. *Let S be a product-quotient surface of general type. Let $\pi: S \rightarrow X$ be the minimal resolution of singularities of the quotient model. Assume that $\pi_*^{-1}(\Gamma)$ is a (-1) -curve in S and let $x \in \text{Sing}(X)$ be a point of type $\frac{1}{n}(1, a)$, with $\frac{n}{a} = [b_1, \dots, b_r]$. Consider the map ν in diagram (3). Then*

- i) $\#\nu^{-1}(x) \leq 1$, if $a = n - 1$;
- ii) $\#\nu^{-1}(x) \leq \sum_{\{b_i \geq 4\}} (b_i - 3) + \#\{i : b_i = 3\}$, if $a \neq n - 1$.

Proof. Note that, since we are assuming $\pi_*^{-1}(\Gamma)$ smooth, $\nu = \pi|_{\pi_*^{-1}(\Gamma)}$.

Let D_i be i -th curve in the resolution graph of x : D_i is smooth, rational with $D_i^2 = -b_i$, whence $K_S D_i = b_i - 2$. We set $d_i := D_i \cdot \pi_*^{-1}(\Gamma)$.

After contracting $\pi_*^{-1}(\Gamma)$, D_i maps to D'_i with $K D'_i = K D_i - d_i$. By remark 4.3, either D'_i is smooth or $K D'_i > 0$. In particular: $d_i \leq \max(1, b_i - 3)$.

If $b_i = 2$ then D_i intersects $\pi_*^{-1}(\Gamma)$ transversally in at most in one point. Moreover, $\pi_*^{-1}(\Gamma)$ can't intersect two D_j with selfintersection -2 , since this would produce, after contracting $\pi_*^{-1}(\Gamma)$, two intersecting (-1) - curves which is impossible on a surface of general type.

Therefore, if $a = n - 1$, $\pi_*^{-1}(\Gamma)$ intersects the whole Hirzebruch-Jung string in at most one point. This shows part i).

In general,

$$\begin{aligned}
 (4) \quad \# \nu^{-1}(x) &\leq \pi_*^{-1}(\Gamma) \left(\sum D_i \right) = \\
 &= \pi_*^{-1}(\Gamma) \left(\sum_{\{b_i \geq 4\}} D_i \right) + \pi_*^{-1}(\Gamma) \left(\sum_{\{b_i = 3\}} D_i \right) + \pi_*^{-1}(\Gamma) \left(\sum_{\{b_i = 2\}} D_i \right) \leq \\
 &\leq \sum_{\{b_i \geq 4\}} (b_i - 3) + \#\{i : b_i = 3\} + 1.
 \end{aligned}$$

It remains to show that, for $a \neq n - 1$, the above inequality cannot be an equality.

In fact, if equality holds, there is an i such that D'_i is a (-1) - curve and $\forall j \neq i$ we have

- $K D'_j = 0$, D'_j is smooth, or
- $K D'_j = 1$, D'_j is singular.

D'_i cannot intersect any singular D'_j , otherwise the surface obtained after contracting D'_i would violate remark 4.3. With the same argument we see that D'_i intersects at most one of the smooth D'_j .

In fact, if $r > 1$, D'_i intersects exactly one smooth D'_j , because a Hirzebruch-Jung string is connected. After the contraction of D'_i , D'_j becomes negative with respect to K , whence it can be contracted. Recursively, we contract all curves. It follows that the dual graph of the union of $\pi_*^{-1}(\Gamma)$ with the Hirzebruch-Jung string of the singularity is a tree. By the connectedness of the Hirzebruch-Jung string, $\pi_*^{-1}(\Gamma) \left(\sum D_i \right) = 1$. Therefore $\forall i$, $b_i = 2$ which is equivalent to $a = n - 1$. \square

The following is an immediate consequence of the above.

Corollary 4.5. *With the same hypotheses as in prop. 4.4 we have:*

- i) $\# \nu^{-1}(\frac{1}{n}(1, 1)) \leq \max(1, n - 3)$;
- ii) $\# \nu^{-1}(\frac{1}{n}(1, a)) \leq 1$, for $n \leq 7$, $a \neq 1$.

One possible definition of a rational double point (RDP for short) is the following. For more details we refer to [BPV84].

Definition 4.6. A *rational double point* is a singular point of a surface, such that all the exceptional curves of the minimal resolution of it have selfintersection -2 .

Proposition 4.7. Assume that S is a product-quotient surface of general type and assume that the basket of singularities of the quotient model X is one of the following:

- 1) $\{\frac{1}{n}(1, a), \frac{1}{n}(1, n-a)\}$ with either $n \leq 4$ or $n \leq 7$, $1 \neq a < \frac{n}{2}$;
- 2) at most one point $\frac{1}{n}(1, a)$ with either $n \leq 4$ or $n \leq 7$, $a \neq 1$, and RDPs;
- 3) $\{2 \times \frac{1}{3}(1, 1) + \text{RDPs}\}$, $\{\frac{1}{5}(1, 1), \frac{1}{5}(1, 4)\}$.

Then S is minimal.

Proof. Assume by contradiction that S contains a (-1) -curve E . Then we can apply prop. 4.4 to $\Gamma := \pi_*(E) \subset X$.

1) In this case, by cor. 4.5 and lemma 4.1, γ (cf. diagram 3) has at most two critical values, corresponding to the singular points of X . Therefore Γ_1 is rational, a contradiction.

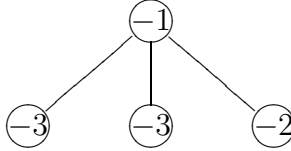
2) Note that E cannot intersect two distinct (-2) - curves. In particular, Γ can pass through at most one rational double point, and has to be smooth in this point.

Therefore, this case is excluded by the same argument as above.

3) We have to treat each basket separately.

$\{2 \times \frac{1}{3}(1, 1) + \text{RDPs}\}$: by corollary 4.5 γ has at most 3 branch points, whence by the above argument it has exactly 3. Therefore Γ passes through both triple points and through one rational double point of X : we have found a configuration of rational curves on S whose dual graph is

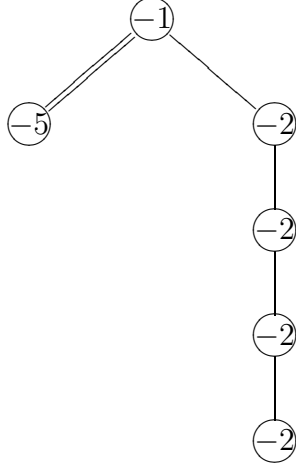
(5)



which cannot occur on a surface of general type because, after contracting the (-1) - and the (-2) -curve, one gets two intersecting (even tangentially) (-1) -curves.

$\{\frac{1}{5}(1, 1), \frac{1}{5}(1, 4)\}$: We get

(6)



Contracting E and the whole H-J string coming from the singularity $\frac{1}{5}(1, 4)$, the image of the (-5) -curve violates rem. 4.3. \square

Theorem 4.8. *The minimal product-quotient surfaces of general type with $p_g = 0$ form 72 families which are listed in tables 1 and 2, and described in appendix A.*

Proof. The case $K^2 = 8$ has been already classified in [BC04], [BCG08].

Running our program for $K^2 \in \{7, 6, 5, 4, 3, 2, 1\}$ we have found the surfaces listed in tables 1 and 2 and one more surface, which we called "the fake Godeaux surface", having $K_S^2 = 1$ and $\pi_1(S) = \mathbb{Z}/6\mathbb{Z}$ (hence cannot be minimal, cf. [Rei78]).

All the other surfaces are minimal by comparing the baskets appearing in tables 1 and 2 with proposition 4.7 (remembering that $\frac{1}{5}(1, 2) = \frac{1}{5}(1, 3)$). \square

5. THE FAKE GODEAUX SURFACE

Our program produces 73 families of product-quotient surfaces of general type with $p_g = 0$ and $K^2 > 0$, and theorem 4.8 shows that 72 of them are families of minimal surfaces.

The 73rd output in the form of tables 1 and 2 is the following

TABLE 4.

| K_S^2 | Sing X | t_1 | t_2 | G | N | $H_1(S, \mathbb{Z})$ | $\pi_1(S)$ |
|---------|--------------|----------|-----------|-------------|---|----------------------|----------------|
| 1 | $1/7, 2/7^2$ | $3^2, 7$ | $2, 4, 7$ | $PSL(2, 7)$ | 1 | \mathbb{Z}_6 | \mathbb{Z}_6 |

More precisely, the computer gives exactly one pair of appropriate orbifold homomorphisms, which is the following.

We see $G = PSL(2, 7)$ as subgroup of \mathfrak{S}_8 generated by $(367)(458), (182)(456)$. Then (note that $\mathbb{T}(a, b, c) \cong \mathbb{T}(c, b, a)$)

$$\begin{array}{ll} \varphi_1: \mathbb{T}(7, 3, 3) \rightarrow G, & \varphi_2: \mathbb{T}(7, 4, 2) \rightarrow G \\ c_1 \mapsto (1824375) & c_1 \mapsto (1658327) \\ c_2 \mapsto (136)(284) & c_2 \mapsto (1478)(2653) \\ c_3 \mapsto (164)(357) & c_3 \mapsto (15)(23)(36)(47). \end{array}$$

As explained in the introduction, choosing three points $p_1, p_2, p_3 \in \mathbb{P}^1$ (as branch points of λ_1), three simple loops γ_i around them with $\gamma_1\gamma_2\gamma_3 = 1$, φ_1 determines the monodromy homomorphism and then C_1 and $\lambda_1: C_1 \rightarrow C_1/G \cong \mathbb{P}^1$. Since the covering is determined by the kernel of the monodromy homomorphism, C_1 and λ_1 do not depend on the choice of the loops.

Since $\text{Aut}(\mathbb{P}^1)$ is 3-transitive, a different choice of the three branch points will give rise to an isomorphic covering.

Therefore in our situation (up to isomorphism) C_1 and λ_1 are unique. The same holds C_2 and λ_2 . Hence the pair (φ_1, φ_2) above determines exactly one product-quotient surface S , which we have called "the fake Godeaux surface".

Note that by remark 4.2 S is a surface of general type.

This section is devoted to the proof of the following

Theorem 5.1. *The fake Godeaux surface S has two (-1) -curves. Its minimal model has $K^2 = 3$.*

We first construct two (-1) -curves on S .

5.1. The rational curve E' . We can choose the branch points p_i of λ_1 and p'_j of λ_2 at our convenience. We set $(p_1, p_2, p_3) = (1, 0, \infty)$, $(p'_1, p'_2, p'_3) = (0, \infty, -\frac{9}{16})$.

Consider the normalization \hat{C}'_1 of the fibre product between λ_1 and the $\mathbb{Z}/3\mathbb{Z}$ -cover $\xi': \mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined by $\xi'(t) = t^3$. We have a diagram

$$\begin{array}{ccc} \hat{C}'_1 & \xrightarrow{\xi'} & C_1 \\ \downarrow \hat{\lambda}'_1 & & \downarrow \lambda_1 \\ \mathbb{P}^1 & \xrightarrow{\xi'} & \mathbb{P}^1 \end{array}$$

where the horizontal maps are $\mathbb{Z}/3\mathbb{Z}$ -covers and the vertical maps are $PSL(2, 7)$ -covers. Note that ξ' branches on p_2, p_3 which have branching index 3 for λ_1 : it follows that $\hat{\xi}'$ is étale.

The branch points of $\hat{\lambda}'_1$ are the three points in $\xi'^{-1}(p_1)$, all with branching index 7.

For C_2 , we take the normalized fibre product between λ_2 and the map $\eta': \mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined by $\eta'(t) = \frac{(t^3-1)(t-1)}{(t+1)^4}$.

Note that η' has degree 4 and factors through the involution $t \mapsto \frac{1}{t}$. Therefore it is the composition of two double covers, say $\eta' = \eta'_1 \circ \eta'_2$. We get the following diagram

$$\begin{array}{ccccc}
 & & \hat{\eta}' & & \\
 & \nearrow & & \searrow & \\
 \hat{C}'_2 & \xrightarrow{\hat{\eta}'_2} & \bar{C}'_2 & \xrightarrow{\hat{\eta}'_1} & C_2 \\
 \downarrow \hat{\lambda}'_2 & & \downarrow \bar{\lambda}'_2 & & \downarrow \lambda_2 \\
 \mathbb{P}^1 & \xrightarrow{\eta'_2} & \mathbb{P}^1 & \xrightarrow{\eta'_1} & \mathbb{P}^1 \\
 & \searrow & & \nearrow & \\
 & & \eta' & &
 \end{array}$$

where the horizontal maps are $\mathbb{Z}/2\mathbb{Z}$ -covers and the vertical maps are $PSL(2, 7)$ -covers.

A straightforward computation shows that η'_1 branches only on p'_2, p'_3 and therefore $\hat{\eta}'_1$ is étale.

The branch points of $\bar{\lambda}'_2$ are the two points in $(\eta'_1)^{-1}(p'_1)$ with branching index 7, and the point $(\eta'_1)^{-1}(p'_2)$ with branching index 2.

A similar computation shows that the branch points of η'_2 are $(\eta'_1)^{-1}(p'_2)$ and a point $q' \in (\eta'_1)^{-1}(p'_1)$: $\hat{\eta}'_2$ branches on the 24 points of $(\bar{\lambda}'_2)^{-1}(q')$.

The branch points of $\hat{\lambda}'_2$ are the three points of $(\eta')^{-1}(p'_1)$, each with branching index 7.

Lemma 5.2. *$(\hat{C}'_1, \hat{\lambda}'_1)$ and $(\hat{C}'_2, \hat{\lambda}'_2)$ are isomorphic as Galois covers of \mathbb{P}^1 .*

Proof. By construction they have the same group $G = PSL(2, 7)$ and the same branch points, the third roots of 1, each with branching index 7.

We ask the computer for all appropriate orbifold homomorphisms $\varphi: \mathbb{T}(7, 7, 7) \rightarrow PSL(2, 7)$ modulo automorphisms (i.e., inner automorphisms of $PSL(2, 7)$ and Hurwitz moves, cf. [BC04]). The computer finds two possibilities, returned as the sequence $[\varphi(c_1), \varphi(c_2), \varphi(c_3)]$.

```

> FindCurves({* 7^^3 *}, PSL(2,7));
{
  [
    (1, 7, 8, 4, 6, 2, 3),
    (1, 5, 4, 6, 8, 7, 3),
    (1, 2, 6, 8, 4, 5, 3)
  ],
  [

```

$(1, 5, 8, 2, 3, 4, 6),$
 $(1, 6, 3, 4, 7, 5, 8),$
 $(1, 5, 7, 3, 4, 2, 8)$
 $] \}$

There are two conjugacy classes of elements of order 7 in $PSL(2, 7)$. In both sequences the three entries belong to the same conjugacy class, whereas (1784623) is not conjugate to (1582346).

We note the following elementary, but crucial fact: let $\varphi: \mathbb{T} \rightarrow G$ be an appropriate orbifold homomorphism such that all $\varphi(c_i)$ belong to the same conjugacy class \mathcal{C} . Let φ' be an appropriate orbifold homomorphism which is equivalent to φ under the equivalence relation generated by inner automorphisms of G and by Hurwitz moves. Then $\varphi'(c_i) \in \mathcal{C}$ for every i .

We denote by $\hat{\varphi}_i$ an appropriate orbifold homomorphism associated to $\hat{\lambda}'_i$.

To prove the lemma it suffices now to show that there exist i, j such that $\hat{\varphi}_1(c_i)$ is conjugate to $\hat{\varphi}_2(c_j)$.

By construction, the branch points of $\hat{\lambda}'_1$ are the three points in $\xi'^{-1}(p_1)$, and they are all regular points of ξ' . This implies that $\hat{\varphi}_1(c_i)$ is conjugate to $\varphi_1(c_1) = (1824375)$.

Similarly, the branch points of $\hat{\lambda}'_2$ are the three points of $(\eta')^{-1}(p'_1)$, two of them are regular points of η' . This implies that two of the $\hat{\varphi}_2(c_i)$ are conjugate to $\varphi_2(c_1) = (1658327)$, which is conjugate to (1824375). \square

Consider the curve $\hat{C}' := \hat{C}'_1 = \hat{C}'_2$. By Hurwitz' formula it is a smooth curve of genus $1 + \frac{168}{2}(-2 + 3\frac{6}{7}) = 49$ on which we have an action of $G = PSL(2, 7)$, an action of $\mathbb{Z}/3\mathbb{Z}$, and an action of $\mathbb{Z}/2\mathbb{Z}$ (given by $\hat{\eta}'_2$). Note that the last two commute with the first (in fact, these two generate an action of \mathfrak{S}_3 on \hat{C}' , just look at the induced action on $\hat{C}'/G = \mathbb{P}^1$, and how they permute the third roots of 1, so we have an explicit faithful action of $PSL(2, 7) \times \mathfrak{S}_3$ on \hat{C}').

We have then a divisor $C' := (\hat{\xi}', \hat{\eta}')(\hat{C}') \subset C_1 \times C_2$ which is G -invariant, and the quotient is a rational curve $\hat{C}'/G \cong \mathbb{P}^1 \xrightarrow{e'} D'$ contained in the quotient model X of the fake Godeaux surface S .

Proposition 5.3. *D' has an ordinary double point at the singular point $\frac{1}{7}(1, 1)$, and contains one more singular point of X .*

Let E' be the strict transform of D' on S , let E_7 be the exceptional divisor over the singular point of type $\frac{1}{7}(1, 1)$, E_2, E_4 be the exceptional divisors over the other singular point contained in D' , with $E_d^2 = -d$.

Then $E'E_7 = 2$, $E'E_4 = 1$, $E'E_2 = 0$ and E' is numerically equivalent to $\pi^*D' - \frac{1}{7}(2E_7 + E_2 + 2E_4)$.

Moreover, E' is a smooth rational curve with selfintersection -1 .

Proof. The composition of e' with λ is the map (ξ', η') , which is birational onto its image. Therefore e' is also birational, and D' is singular at most over the singular points of $(\xi', \eta')(\mathbb{P}^1) =: R'$.

Consider the point $(1, 0) \in \mathbb{P}^1 \times \mathbb{P}^1$; it is the image of the third roots of 1 under the map (ξ', η') , so R' has a triple point z there.

The points of \hat{C}' lying over z are exactly the 72 points with nontrivial stabilizer for the action of G , divided in 3 orbits, one for each branch of the triple point z of R' .

Choose a branch, let $P \in \hat{C}'$ be one of the 24 points in the corresponding orbit. Since $\hat{\xi}'$ is étale, the map $(\hat{\xi}', \hat{\eta}')$ is a local diffeomorphism near P . $\hat{\lambda}'_2(P)$ is one of the three branch points of $\hat{\lambda}'_2$ (depending only on the chosen branch of the singular point z) one of which is of ramification for η'_2 , two are not.

In the latter case, both $\hat{\xi}'$, $\hat{\eta}'$ are local diffeomorphisms, equivariant for the action of the stabilizer of P . It follows that there are local coordinates in $C_1 \times C_2$ such that the corresponding branch of C' is $\{x = y\}$ and the group acts as $(x, y) \mapsto (e^{\frac{2\pi i}{7}}x, e^{\frac{2\pi i}{7}}y)$: we have then two branches of D' through the singular point $\frac{1}{7}(1, 1)$.

If instead $\hat{\lambda}'_2(P)$ is a ramification point of η'_2 , the local equation of the branch is $\{x^2 = y\}$ and the action is $(x, y) \mapsto (e^{\frac{2\pi i}{7}}x, e^{\frac{4\pi i}{7}}y)$: the corresponding branch of D' passes through a point $\frac{1}{7}(1, 2)$ and a local computation shows that its strict transform intersects transversally the (-4) -curve and does not intersect the (-2) -curve.

We have computed how E' intersects the E_d , the claim on the numerical equivalence follows by standard intersection arguments.

Then

$$K_{C_1 \times C_2} C' = 18 \cdot 4 + 3 \cdot 32 = 168$$

$$\Rightarrow K_X \cdot D' = 1 \Rightarrow K_S \cdot E' = K_X \cdot D' - \frac{1}{7}(2K_S E_7 + K_S E_2 + 2K_S E_4) = -1$$

Since S is of general type and E' is irreducible with $K_S E' < 0$, by remark 4.3 E' is smooth. This concludes the proof. \square

5.2. The rational curve E'' . The construction is similar to the previous one. We change the choice of the branch points, here $(p_1, p_2, p_3) = (0, \frac{i}{3\sqrt{3}}, -\frac{i}{3\sqrt{3}})$, $(p'_1, p'_2, p'_3) = (1, \infty, 0)$.

We define three maps $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ as follows: $\xi''_2(t) = \frac{2t}{t^2+1}$, $\xi''_1(t) = \frac{t^2-1}{t^3-9t}$, $\eta''(t) = t^4$.

Note that ξ_2'' is the quotient by the involution $t \mapsto \frac{1}{t}$, ξ_1'' is the $\mathbb{Z}/3\mathbb{Z}$ -cover given by $t \mapsto \frac{t-3}{t+1}$, and η'' is the $\mathbb{Z}/4\mathbb{Z}$ -cover given by $t \mapsto it$.

By taking normalized fibre products as in the previous case, we get two commutative diagrams:

$$\begin{array}{ccccc}
 & & \xi'' & & \\
 & \nearrow \hat{\xi}_2'' & & \nwarrow \hat{\xi}_1'' & \\
 \hat{C}_1''' & \xrightarrow{\quad} & \bar{C}_1''' & \xrightarrow{\quad} & C_1 \\
 \downarrow \hat{\lambda}_1'' & & \downarrow \bar{\lambda}_1'' & & \downarrow \lambda_1 \\
 \mathbb{P}^1 & \xrightarrow{\quad \xi_2'' \quad} & \mathbb{P}^1 & \xrightarrow{\quad \xi_1'' \quad} & \mathbb{P}^1 \\
 & \nwarrow \xi'' & & \nearrow \xi'' & \\
 & & \eta'' & & \\
 \hat{C}_2''' & \xrightarrow{\quad} & C_2 & & \\
 \downarrow \hat{\lambda}_2'' & & \downarrow \lambda_2 & & \\
 \mathbb{P}^1 & \xrightarrow{\quad \eta'' \quad} & \mathbb{P}^1 & &
 \end{array}$$

where the vertical maps are $PSL(2, 7)$ -covers and the horizontal maps are cyclic covers.

Note that η'' branches on p'_2, p'_3 , ξ_1'' on p_2, p_3 , ξ_2'' on $\{\pm 1\} \subset (\xi_1'')^{-1}(p_1)$.

Lemma 5.4. *$(\hat{C}_1'', \hat{\lambda}_1'')$ and $(\hat{C}_2'', \hat{\lambda}_2'')$ are isomorphic as Galois cover of \mathbb{P}^1 .*

Proof. Arguing as in the previous case, we see that $\hat{\lambda}_1''$ is a $PSL(2, 7)$ -cover with the four branch points $\xi''^{-1}(p_1)$, each of branching index 7, and $\hat{\lambda}_2''$ is a $PSL(2, 7)$ -cover with the four branch points $\eta''^{-1}(p'_1)$, each of branching index 7. Indeed, $\xi''^{-1}(p_1) = \hat{\lambda}_2''$ is the set of the fourth roots of unity.

We denote by $\hat{\varphi}_i$ an appropriate orbifold homomorphism associated to $\hat{\lambda}_i''$. Arguing as in the proof of lemma 5.2 we see that for all i , $\hat{\varphi}_1(c_i)$ is conjugate to $\varphi_1(c_1)$ or to $\varphi_1(c_1)^2$. Similarly, $\hat{\varphi}_2(c_i)$ is conjugate to $\varphi_2(c_1)$. Since the three elements $\varphi_1(c_1)$, $\varphi_1(c_1)^2$ and $\varphi_2(c_1)$ are conjugate in G , all $\hat{\varphi}_j(c_i)$ are conjugate to $\varphi_1(c_1) = (1824375)$.

The following computation shows that there are two equivalence classes of appropriate orbifold homomorphisms $\varphi: \mathbb{T}(7, 7, 7, 7) \rightarrow PSL(2, 7)$, distinguished by the following feature: in one class the $\varphi(c_i)$ are always pairwise distinct.

```

> #FindCurves({* 7^4 *}, PSL(2,7));
8
> L:={@ @};
> for seq in FindCurves({* 7^4 *}, PSL(2,7)) do test:= true;
for> for g in seq do

```



```

for|for> if not IsConjugate(PSL(2,7),g,PSL(2,7)!(1,8,2,4,3,7,5))
    then test:=false; break g;
for|for|if> end if;
for|for> end for;
for> if test then Include(~L,seq);
for|if> end if;
for> end for;
> #L;
2
> for k in [1..#L] do
for> M:={@ @};
for> for seq in HurwitzOrbit(L[k]) do
for|for> for i in [1..3] do for j in [i+1..4] do
for|for|for|for> if seq[i] eq seq[j] then Include(~M, seq);break i;
for|for|for|for|if> end if; end for; end for;end for;
for> #M;
for> end for;
0
840
>

```

We need to show that $\hat{\varphi}_1$ and $\hat{\varphi}_2$ belong to same class, in fact to the second.

$\hat{\varphi}_2$: Consider the map η'' , and choose as base point for $\pi_1(\mathbb{P}^1 \setminus \{\pm 1, \pm i\}, p)$ a point $p = \epsilon$, $\epsilon \in \mathbb{R}$, $0 < \epsilon \ll 1$.

We define the following geometric loops with starting point ϵ :

- γ_1 moves on the real axis from ϵ to $1 - \epsilon$, then makes a circle counterclockwise around 1, and moves back on the real axis to ϵ .
- $\gamma_2 = \alpha(i\gamma_1)\alpha^{-1}$ where α is a quarter of a circle around 0 from ϵ to $i\epsilon$.
- $\gamma_3 = \beta(-\gamma_1)\beta^{-1}$ where β is a half circle around 0 from ϵ to $-\epsilon$.
- γ_4 is a similarly defined loop around $-i$.

Then $\gamma_1 \cdots \gamma_4 = 1$. Now it is easy to see (since 0 is a branch point of branching index 2 for λ_2) that the image of γ_1 in G is the same as the image of γ_3 (and the image of γ_2 is the same as the image γ_4).

$\hat{\varphi}_1$: Let $\gamma_1, \gamma_\infty, \gamma_{-1}$ be geometric loops with base point $p = 0$, γ_j around j , $\gamma_1\gamma_\infty\gamma_{-1} = 1$ in $\pi_1(\mathbb{P}^1 \setminus \{\pm 1, \infty\})$. Then we can find geometric loops $\mu_1, \mu_i, \mu_{-1}, \mu_{-i}$ with base point 0, μ_j around j , $\mu_1\mu_i\mu_{-1}\mu_{-i} = 1$ in $\pi_1(\mathbb{P}^1 \setminus \{\pm 1, \pm i\})$, such that $\xi_2'' \circ \mu_i = \gamma_\infty$, $\xi_2'' \circ \mu_{-i} = \gamma_1^7\gamma_\infty$, which have the same image in G . \square

The curve $\hat{C}''' := \hat{C}_1''' = \hat{C}_2'''$ is a smooth curve of genus $1 + \frac{168}{2}(-2 + 4\frac{6}{7}) = 121$ with an action of $\text{PSL}(2,7)$ (in fact, of $\text{PSL}(2,7) \times D_4$, where D_4 is the dihedral group of

order 8). The divisor $C'' := (\hat{\xi}'', \hat{\eta}'')(\hat{C}'') \subset C_1 \times C_2$ is G -invariant, and the quotient is a rational curve $\hat{C}''/G \cong \mathbb{P}^1 \xrightarrow{e''} D'' \subset X$. Note that $\lambda \circ e'' = (\xi'', \eta'')$, which is birational. Therefore e'' is also birational.

Proposition 5.5. *D'' has an ordinary double point at the singular point $\frac{1}{7}(1, 1)$ and contains both the other singular points of X .*

Let E'' be the strict transform of D'' on S , let E_7 be the exceptional divisor over the singular point of type $\frac{1}{7}(1, 1)$, E_2, E_4, E'_2, E'_4 be the other exceptional divisors, with $E_d^2 = (E'_d)^2 = -d$, $E_2E_4 = E'_2E'_4 = 1$.

Then $E''E_7 = 2$, $E''E_4 = E''E'_4 = 1$, $E''E_2 = E''E'_2 = 0$ and E'' is numerically equivalent to $D'' - \frac{1}{7}(2E_7 + E_2 + 2E_4 + E'_2 + 2E'_4)$.

Moreover, E'' is a smooth rational curve with selfintersection -1 .

Proof. Consider the point $(0, 1)$, quartuple point of R' . The points of \hat{C}'' dominating it are exactly the 96 points with nontrivial stabilizer for the action of G , divided in 4 orbits, one for each branch of R' .

Choose a branch of the quartuple point, let $P \in \hat{C}''$ be one of the 24 points in the corresponding orbit above it. Since $\hat{\eta}''$ is étale, the map $(\hat{\xi}'', \hat{\eta}'')$ is a local diffeomorphism near P . $\hat{\lambda}_1''(P)$ is one of the branch points of $\hat{\lambda}_1''$ (depending only on the choosen branch), two of which are ramification points of ξ_2'' , two are not.

In the latter case, arguing as in the proof of proposition 5.3, the corresponding branch of D'' passes through the singular point of type $\frac{1}{7}(1, 1)$ and is smooth there. Instead, in the first case, it passes through a point of type $\frac{1}{7}(1, 2)$, and its strict transform intersects transversally the (-4) -curve and does not intersect the (-2) -curve.

It follows that $E''E_7 = 2$, $E''E_2 = E''E'_2$, $E''(E_4 + E'_4) = 2$. We still do not know whether D'' passes through both singular points $\frac{1}{7}(1, 2)$ (equivalently $E''E_4 = E''E'_4 = 1$), or misses one of them and passes twice through the other.

Then

$$K_{C_1 \times C_2} C'' = 18 \cdot 4 + 6 \cdot 32 = 264$$

$$\Rightarrow K_X \cdot D'' = \frac{11}{7} \Rightarrow K_S \cdot E'' = \frac{11}{7} - 2\frac{5}{7} - 2\frac{4}{7} = -1.$$

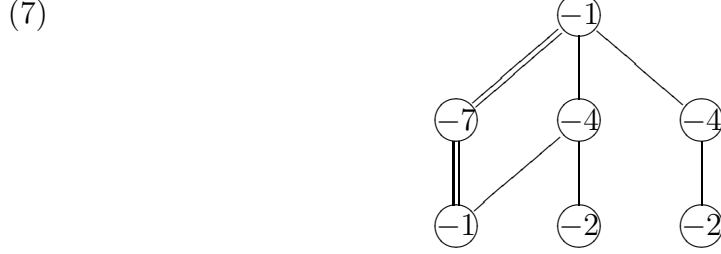
Since S is of general type and E'' is irreducible with $K_S E'' < 0$, by remark 4.3, E'' is smooth. This proves that E'' is a rational (-1) -curve.

If $E''E_4 = 2$ or $E''E'_4 = 2$, after contracting E'' , we get a contradiction to remark 4.3.

Therefore, $E''E_4 = E''E'_4 = 1$. □

Corollary 5.6. *Let $\pi': S \rightarrow S'$ be the blow down of E' and E'' . Then S' is minimal.*

Proof. Since S is of general type, E' and E'' are disjoint. We have a configuration of rational curves on S , whose dual graph is the following:



After contracting E' and E'' the induced configuration consists of a singular (-3) -curve, three smooth (-2) -curves and a smooth (-3) -curve.

Assume that there is a smooth rational curve E''' with selfintersection (-1) on S' . By the same arguments as in prop. 4.4 E''' can intersect only one of the (-2) -curves (with multiplicity one), and the smooth (-3) -curve (again with multiplicity one). The singular (-3) curve instead cannot intersect E''' , because this would (after contracting E''') give a contradiction to remark 4.3.

Let $\Gamma \subset X$ be the rational curve $\pi_* \pi'^{-1} E'''$ on X . Then, by lemma 4.1 the induced map γ (cf. diagram 3) has at most two critical values, and therefore Γ_1 is rational, a contradiction (cf. proof of prop. 4.7). \square

6. SOME REMARKS ABOUT THE COMPUTATIONAL COMPLEXITY

In this short section we will comment on the necessity to use a computer algebra program in this paper, and also on the time and memory that is needed for the various calculation we did.

We use the computer algebra program MAGMA, but our algorithms can be implemented in any other computer algebra program which has a database of finite groups (e.g. GAP4).

The heaviest computational problem we encountered are caused by the first step of the algorithm in section 2: there we compute for each $1 \leq K^2 \leq 7$ the possible baskets of singularities for $(K^2, 1)$. Our algorithm is quite slow, but has a very low memory usage. Indeed, making the algorithm quicker had disastrous effects on the memory usage.

In the following table we report the computation time and memory usage of the script *Baskets* for each K^2 . Almost all computations have been done on a simple workstation with 4GB of RAM.

| K^2 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|-------------|------|------|------|------|------|-------|---------|---------|
| time (s) | 0.00 | 0.00 | 0.01 | 0.11 | 2.16 | 45.99 | 1185.85 | 43316.7 |
| memory (MB) | 7.64 | 7.64 | 7.64 | 7.64 | 7.64 | 8.31 | 9.83 | 18.42 |

An improvement of this algorithm (i.e., to make it faster without substantially increasing the memory usage) would constitute the major step towards extending the results of the present paper to negative values of K^2 . Indeed we give the analogous table for *ExistingSurfaces*.

| K^2 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|-------------|------|--------|--------|--------|--------|--------|--------|--------|
| time (s) | 0.00 | 1811 | 3659 | 5132 | 385 | 8065 | 2632 | 84989 |
| memory (MB) | 7.64 | 119.03 | 119.03 | 119.55 | 118.64 | 120.06 | 120.23 | 397.39 |

ExistingSurfaces first runs *Baskets*. We notice that for the first cases the time requested by this first computation is negligible. For $K^2 = 0$ it is more or less the half.

The other scripts in the main algorithm are quite harmless in time and memory usage.

The computations in sections 3, 5 are neither time nor memory demanding, except for propositions 3.3 and 3.4, where we had to use a SmallGroup Process (and they actually are quite heavy). In fact, we had to run those two computations on a better workstation (32GB of RAM).

The first lasted 192261.54 sec. (53-54 hours) and needed 19102.22 MB, the second lasted 4581.27 sec. and needed 4509.09 MB.

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APPENDIX A. THE MINIMAL PRODUCT-QUOTIENT SURFACES OF GENERAL TYPE WITH $p_g = 0$ AND $K^2 < 8$

In this section we describe all the minimal product-quotient surfaces we have listed in tables 1 and 2, with the exception of the one whose singular model X has at worse canonical singularities (these are already described in [BCG08] and [BCGP08]).

In the sequel we will follow the scheme below:

G : here we write the group G (most of the times as permutation group);

- t_i : here we specify the respective types of the pair of spherical generators of the group G ;
 S_1 : here we list the first set of spherical generators;
 S_2 : here we list the second set of spherical generators;
 H_1 : the first homology group of the surface;
 π_1 : the fundamental group of the surface;

A.1. $K^2 = 5$, **basket** $\{\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)\}$.

A.1.1. *Group* $\mathfrak{S}_4 \times \mathbb{Z}_2$:

- G : $\langle (12), (13), (14), (56) \rangle < \mathfrak{S}_6$;
 t_i : $(3, 2^4)$ and $(6, 4, 2)$;
 S_1 : $(134), (34)(56), (13)(24)(56), (23)(56), (13)(24)(56)$;
 S_2 : $(234)(56), (4321)(56), (14)$;
 H_1 : $\mathbb{Z}_2^2 \times \mathbb{Z}_4$;
 π_1 : the fundamental group of this surface fits in two exact sequences

$$1 \rightarrow \mathbb{Z}^2 \rightarrow \pi_1 \rightarrow D_{2,8,3} \rightarrow 1$$

$$1 \rightarrow \mathbb{Z}^2 \rightarrow \pi_1 \rightarrow Q(16) \rightarrow 1$$

where $Q(16)$ is the generalized quaternion group of order 16.

The normal subgroups of index 16 of π_1 on the left have minimal index among the normal subgroups of π_1 with free abelianization. Let us recall that $D_{2,8,3}$ is the group $\langle x, y | x^2, y^8, xyx^{-1}y^{-3} \rangle$ and $Q(16)$ is the group $\langle x, y | x^8, x^4y^{-2}, yxy^{-1}x \rangle$.

A.1.2. *Group* \mathfrak{S}_4 .

- G : \mathfrak{S}_4 ;
 t_i : $(3, 2^4)$ and $(4^2, 3)$;
 S_1 : $(124), (23), (24), (14), (13)$;
 S_2 : $(1243), (1234), (123)$;
 H_1 : $\mathbb{Z}_2^2 \times \mathbb{Z}_8$;
 π_1 : the fundamental group of this surface fits in an exact sequence

$$1 \rightarrow \mathbb{Z}^2 \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1$$

and the normal subgroup of index 8 of π_1 on the left has minimal index among the normal subgroups of π_1 with free abelianization.

A.1.3. *Group $\mathfrak{S}_4 \times \mathbb{Z}_2$:*

G : $\langle (12), (13), (14), (56) \rangle < \mathfrak{S}_6$;
 t_i : $(3, 2^3)$ and $(6, 4^2)$;
 S_1 : $(143), (12), (24)(56), (12)(34)(56)$;
 S_2 : $(134)(56), (1342)(56), (1234)$;
 H_1 : $\mathbb{Z}_2^2 \times \mathbb{Z}_8$;
 π_1 : the fundamental group fits in an exact sequence

$$1 \rightarrow \mathbb{Z}^2 \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1$$

and the normal subgroup of index 8 of π_1 on the left has minimal index among the normal subgroups of π_1 with free abelianization.

A.1.4. *Group \mathfrak{S}_5 .*

G : \mathfrak{S}_5 ;
 t_i : $(6, 5, 2)$ and $(4^2, 3)$;
 S_1 : $(13)(245), (14253), (34)$;
 S_2 : $(4321), (1534), (235)$;
 H_1 : \mathbb{Z}_8 ;
 π_1 : $D_{8,5,-1} = \langle x, y | x^8, y^5, xyx^{-1}y \rangle$.

A.1.5. *Group \mathfrak{A}_5 .*

G : \mathfrak{A}_5 ;
 t_i : $(3, 2^3)$ and $(5^2, 3)$;
 S_1 : $(152), (14)(23), (23)(45), (14)(25)$;
 S_2 : $(15423), (13425), (254)$;
 H_1 : $\mathbb{Z}_2 \times \mathbb{Z}_{10}$;
 π_1 : $\mathbb{Z}_5 \times Q_8$, where Q_8 is the quaternion group $\langle x, y | x^4, x^2y^{-2}, xyx^{-1}y \rangle$.

A.1.6. *Group $\mathbb{Z}_2^4 \rtimes \mathfrak{S}_3$:* this is the semidirect product obtained by letting (12) and

(123) act on \mathbb{Z}_2^4 respectively as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

G : $\langle x_1, x_2, x_3, x_4, y_2, y_3 | x_i^2, y_i^2, [x_i, x_j], (y_2y_3)^2, y_2x_{2i-1}y_2x_{2i}, y_3^{-1}x_{2i-1}y_3x_{2i} \rangle$;
 t_i : $(3, 2^3)$ and $(4^2, 3)$;
 S_1 : $y_3x_1, y_2y_3^2x_3, x_1x_3x_4, y_2y_3x_4$;
 S_2 : $y_2x_1x_4, y_2y_3x_1, y_3^2x_1x_3$;
 H_1 : $\mathbb{Z}_2 \times \mathbb{Z}_8$;
 π_1 : the fundamental group of this surface fits in an exact sequences

$$1 \rightarrow H \rightarrow \pi_1 \rightarrow D_{8,4,3} \rightarrow 1$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = D_{8,4,3}$ but the computer could not solve the problem. Recall that $D_{8,4,3}$ is the group $\langle x, y | x^8, y^4, xyx^{-1}y^{-3} \rangle$.

A.1.7. *Group \mathfrak{A}_5 .*

G : \mathfrak{A}_5 ;
 t_i : $(3, 2^3)$ and $(5^2, 3)$;
 S_1 : $(152), (14)(23), (23)(45), (14)(25)$;
 S_2 : $(14235), (15243), (123)$;
 H_1 : $\mathbb{Z}_2 \times \mathbb{Z}_{10}$;
 π_1 : $\mathbb{Z}_2 \times \mathbb{Z}_{10}$.

A.2. $K^2 = 4$, **basket** $\{2 \times \frac{1}{5}(1, 2)\}$.

A.2.1. *Group \mathfrak{A}_5 .*

G : \mathfrak{A}_5 ;
 t_i : $(5, 2^3)$ and $(5, 3^2)$;
 S_1 : $(13245), (12)(34), (15)(23), (14)(35)$;
 S_2 : $(13542), (123), (345)$;
 H_1 : $\mathbb{Z}_2 \times \mathbb{Z}_6$;
 π_1 : $\mathbb{Z}_2 \times \mathbb{Z}_6$.

A.2.2. *Group $\mathbb{Z}_2^4 \rtimes D_5$* : this is the semidirect product obtained by letting a symmetry

and a rotation of D_5 act on \mathbb{Z}_2^4 respectively as $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

G :

$$\langle x_1, x_2, x_3, x_4, y_2, y_5 | \quad x_i^2, y_i^i, [x_i, x_j], (y_2 y_5)^2, \\ y_2 x_1 y_2 x_1 x_2, y_2 x_2 y_2 x_2, y_2 x_3 y_2 x_1 x_2 x_4, y_2 x_4 y_2 x_1 x_3, \\ y_5^{-1} x_1 y_5 x_1 x_2, y_5^{-1} x_2 y_5 x_2 x_3, y_5^{-1} x_3 y_5 x_3 x_4, y_5^{-1} x_4 y_5 x_1 \rangle$$

t_i : $(5, 4^2)$ and $(5, 4, 2)$;
 S_1 : $y_5^2 x_1, y_2 y_5^2 x_2 x_4, y_2 x_4$;
 S_2 : $y_5 x_2 x_3, y_2 y_5 x_1 x_2 x_3 x_4, y_2 x_1 x_3 x_4$;
 H_1 : \mathbb{Z}_8 ;
 π_1 : the fundamental group fits in an exact sequences

$$1 \rightarrow H \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.2.3. *Group $\mathbb{Z}_2^4 \rtimes D_5$:*

G : as above;

t_i : $(5, 4^2)$ and $(5, 4, 2)$;

S_1 : $y_5^3 x_1 x_4, y_2 x_3, y_2 y_5^2 x_2 x_4$;

S_2 : $y_5^4 x_1 x_2 x_3, y_2 x_2 x_4, y_2 y_5$;

H_1 : \mathbb{Z}_8 ;

π_1 : the fundamental group fits in an exact sequences

$$1 \rightarrow H \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.2.4. *Group $\mathbb{Z}_2^4 \rtimes D_5$:*

G : as above;

t_i : $(5, 4^2)$ and $(5, 4, 2)$;

S_1 : $y_5^3 x_1 x_4, y_2 x_3, y_2 y_5^2 x_2 x_4$;

S_2 : $y_5 x_2 x_3, y_2 y_5 x_1 x_2 x_3 x_4, y_2 x_1 x_3 x_4$;

H_1 : \mathbb{Z}_8 ;

π_1 : the fundamental group fits in an exact sequences

$$1 \rightarrow H \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.2.5. *Group \mathfrak{A}_6 .*

G : \mathfrak{A}_6 ;

t_i : $(5, 4, 2)$ and $(5, 3^2)$;

S_1 : $(14623), (13)(2564), (12)(56)$;

S_2 : $(14562), (134)(265), (243)$;

H_1 : \mathbb{Z}_6 ;

π_1 : \mathbb{Z}_6 .

A.3. $K^2 = 3$, **basket** $\{\frac{1}{5}(1, 1) + \frac{1}{5}(1, 4)\}$.

A.3.1. *Group \mathfrak{A}_5 .*

G : \mathfrak{A}_5 ;
 t_i : $(5, 2^3)$ and $(5, 3^2)$;
 S_1 : $(14235), (23)(45), (13)(45), (14)(35)$;
 S_2 : $(13542), (123), (345)$;
 H_1 : $\mathbb{Z}_2 \times \mathbb{Z}_6$;
 π_1 : $\mathbb{Z}_2 \times \mathbb{Z}_6$.

A.3.2. *Group $\mathbb{Z}_2^4 \rtimes D_5$: this is the semidirect product obtained by letting a symmetry*

and a rotation of D_5 act on \mathbb{Z}_2^4 respectively as $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

G :

$\langle x_1, x_2, x_3, x_4, y_2, y_5 \mid$
 $x_i^2, y_i^i, [x_i, x_j], (y_2 y_5)^2,$
 $y_2 x_1 y_2 x_1 x_2, y_2 x_2 y_2 x_2, y_2 x_3 y_2 x_1 x_2 x_4, y_2 x_4 y_2 x_1 x_3,$
 $y_5^{-1} x_1 y_5 x_1 x_2, y_5^{-1} x_2 y_5 x_2 x_3, y_5^{-1} x_3 y_5 x_3 x_4, y_5^{-1} x_4 y_5 x_1 \rangle;$

t_i : $(5, 4^2)$ and $(5, 4, 2)$;
 S_1 : $y_5^2 x_1, y_2 y_5^2 x_2 x_4, y_2 x_4$;
 S_2 : $y_5^3 x_1 x_3, y_2 y_5^3 x_4, y_2 x_1 x_3 x_4$;
 H_1 : \mathbb{Z}_8 ;
 π_1 : the fundamental group fits in an exact sequences

$$1 \rightarrow H \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.3.3. *Group $\mathbb{Z}_2^4 \rtimes D_5$:*

G : as above;
 t_i : $(5, 4^2)$ and $(5, 4, 2)$;
 S_1 : $y_5^3 x_1 x_4, y_2 x_3, y_2 y_5^2 x_2 x_4$;
 S_2 : $y_5^3 x_2 x_4, y_2 y_5^2 x_1 x_4, y_2 y_5^4 x_1 x_2 x_4$;
 H_1 : \mathbb{Z}_8 ;
 π_1 : the fundamental group fits in an exact sequences

$$1 \rightarrow H \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.3.4. *Group $\mathbb{Z}_2^4 \rtimes D_5$:*

G : as above;
 t_i : $(5, 4^2)$ and $(5, 4, 2)$;
 S_1 : $y_5^3 x_1 x_4, y_2 x_3, y_2 y_5^2 x_2 x_4$;
 S_2 : $y_5^3 x_1 x_3, y_2 y_5^3 x_4, y_2 x_1 x_3 x_4$;
 H_1 : \mathbb{Z}_8 ;
 π_1 : the fundamental group fits in an exact sequences

$$1 \rightarrow H \rightarrow \pi_1 \rightarrow \mathbb{Z}_8 \rightarrow 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_8$ but the computer could not solve the problem.

A.3.5. *Group \mathfrak{A}_6 .*

G : \mathfrak{A}_6 ;
 t_i : $(5, 4, 2)$ and $(5, 3^2)$;
 S_1 : $(14623), (13)(2564), (12)(56)$;
 S_2 : $(15342), (164), (135)(246)$;
 H_1 : \mathbb{Z}_6 ;
 π_1 : \mathbb{Z}_6 .

A.4. $K^2 = 3$, **basket** $\{2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)\}$.

A.4.1. *Group $\mathfrak{S}_4 \times \mathbb{Z}_2$:*

G : $\langle (12), (13), (14), (56) \rangle < \mathfrak{S}_6$;
 t_i : $(4, 3, 2^2)$ and $(6, 4, 2)$;
 S_1 : $(1234), (234), (13)(24)(56), (34)(56)$;
 S_2 : $(234)(56), (4321)(56), (14)$;
 H_1 : $\mathbb{Z}_2 \times \mathbb{Z}_4$;
 π_1 : $\mathbb{Z}_2 \times \mathbb{Z}_4$.

A.5. $K^2 = 2$, **basket** $\{2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)\}$.

A.5.1. *Group $\mathfrak{A}_4 \times \mathbb{Z}_2$:*

G : $\langle (123), (12)(34), (56) \rangle < \mathfrak{S}_6$;
 t_i : $(6^2, 2)$ and $(3^2, 2^2)$;
 S_1 : $(132)(56), (142)(56), (13)(24)$;
 S_2 : $(234), (123), (13)(24)(56), (14)(23)(56)$;
 H_1 : \mathbb{Z}_2^2 ;
 π_1 : Q_8 .

A.5.2. *Group \mathfrak{S}_4 :*

G : \mathfrak{S}_4 ;
 t_i : $(4^2, 3)$ and $(3^2, 2^2)$;
 S_1 : $(123), (134), (12), (24)$;
 S_2 : $(1234), (1243), (124)$;
 H_1 : \mathbb{Z}_8 ;
 π_1 : \mathbb{Z}_8 .

A.5.3. *Group $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$:* this is the semidirect product obtained by letting a generator of \mathbb{Z}_3 act on \mathbb{Z}_5^2 as $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

G : $\langle x_1, x_2, y | x_i^5, [x_1, x_2], y^3, y^{-1}x_1^{-1}yx_1x_2^2, y^{-1}x_2^{-1}yx_1x_2^3 \rangle$;
 t_i : both $(5, 3^2)$;
 S_1 : $x_1^3x_2^2, y^2x_1^3x_2^4, y$;
 S_2 : $x_1^3, yx_1, y^2x_1^4x_2^2$;
 H_1 : \mathbb{Z}_5 ;
 π_1 : the fundamental group fits in an exact sequences

$$1 \rightarrow H \rightarrow \pi_1 \rightarrow \mathbb{Z}_5 \rightarrow 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_5$ but the computer could not solve the problem.

A.5.4. *Group $\mathbb{Z}_5^2 \rtimes \mathbb{Z}_3$:*

G : as above
 t_i : both $(5, 3^2)$;
 S_1 : $x_1^3x_2^2, y^2x_1^3x_2^4, y$;
 S_2 : $x_1^4x_2^3, yx_1x_2, y^2x_1^4x_2^3$;
 H_1 : \mathbb{Z}_5 ;
 π_1 : the fundamental group fits in an exact sequences

$$1 \rightarrow H \rightarrow \pi_1 \rightarrow \mathbb{Z}_5 \rightarrow 1.$$

where H is a group with a complicated presentation whose abelian quotient is trivial. We conjecture $H = \{1\}$ and $\pi_1 = \mathbb{Z}_5$ but the computer could not solve the problem.

A.5.5. *Group \mathfrak{A}_5 .*

G : \mathfrak{A}_5 ;
 t_i : $(5, 3^2)$ and $(3, 2^3)$;
 S_1 : $(13542), (123), (345)$;
 S_2 : $(152), (14)(23), (23)(45), (14)(25)$;
 H_1 : \mathbb{Z}_2^2 ;
 π_1 : \mathbb{Z}_2^2 .

A.6. $K^2 = 2$, **basket** $\{2 \times \frac{1}{2}(1, 1) + \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)\}$.

A.6.1. *Group $PSL(2, 7)$:*

G : $\langle (34)(56), (123)(457) \rangle < \mathfrak{S}_7$;
 t_i : $(7, 4, 2)$ and $(4, 3^2)$;
 S_1 : $(1436275), (14)(2357), (36)(45)$;
 S_2 : $(1236)(47), (245)(376), (164)(257)$;
 H_1 : \mathbb{Z}_3 ;
 π_1 : \mathbb{Z}_3 .

A.6.2. *Group $PSL(2, 7)$:*

G : $\langle (34)(56), (123)(457) \rangle < \mathfrak{S}_7$;
 t_i : $(7, 4, 2)$ and $(4, 3^2)$;
 S_1 : $(1436275), (14)(2357), (36)(45)$;
 S_2 : $(34)(1675), (164)(257), (134)(265)$;
 H_1 : \mathbb{Z}_3 ;
 π_1 : \mathbb{Z}_3 .

A.6.3. *Group \mathfrak{A}_6 .*

G : \mathfrak{A}_6 ;
 t_i : $(5, 4, 2)$ and $(4, 3^2)$;
 S_1 : $(14623), (13)(2564), (12)(56)$;
 S_2 : $(16)(2435), (246), (162)(345)$;
 H_1 : \mathbb{Z}_3 ;
 π_1 : \mathbb{Z}_3 .

A.6.4. *Group \mathfrak{A}_6 .*

G : \mathfrak{A}_6 ;
 t_i : $(5, 4, 2)$ and $(4, 3^2)$;
 S_1 : $(14623), (13)(2564), (12)(56)$;
 S_2 : $(1365)(24), (124)(356), (125)$;
 H_1 : \mathbb{Z}_3 ;
 π_1 : \mathbb{Z}_3 .

A.6.5. *Group \mathfrak{S}_5 .*

G : \mathfrak{S}_5 ;
 t_i : $(5, 4, 2)$ and $(6, 4, 3)$;
 S_1 : $(15432), (1235), (45)$;
 S_2 : $(15)(234), (2453), (153)$;
 H_1 : \mathbb{Z}_3 ;
 π_1 : \mathbb{Z}_3 .

A.6.6. *Group \mathfrak{S}_5 .*

G : \mathfrak{S}_5 ;
 t_i : $(5, 4, 2)$ and $(6, 4, 3)$;
 S_1 : $(15432), (1235), (45)$;
 S_2 : $(14)(235), (1254), (432)$;
 H_1 : \mathbb{Z}_3 ;
 π_1 : \mathbb{Z}_3 .

A.7. $K^2 = 1$, **basket** $\{4 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)\}$.A.7.1. *Group \mathfrak{S}_5 .*

G : \mathfrak{S}_5 ;
 t_i : $(3, 2^3)$ and $(4^2, 3)$;
 S_1 : $(123), (34), (23), (13)(24)$;
 S_2 : $(1234), (1243), (124)$;
 H_1 : \mathbb{Z}_4 ;
 π_1 : \mathbb{Z}_4 .

A.7.2. *Group $PSL(2, 7)$:*

G : $\langle (34)(56), (123)(457) \rangle < \mathfrak{S}_7$;
 t_i : $(7, 3, 2)$ and $(4^2, 3)$;
 S_1 : $(1476532), (164)(235), (26)(47)$;
 S_2 : $(1765)(23), (17)(3645), (236)(475)$;
 H_1 : \mathbb{Z}_2 ;
 π_1 : \mathbb{Z}_2 .

A.7.3. Group $\mathfrak{S}_4 \times \mathbb{Z}_2$:

$G: \langle (12), (13), (14), (56) \rangle < \mathfrak{S}_6$;
 $t_i: (3, 2^3) \text{ and } (6, 4, 2)$;
 $S_1: (134), (13)(24)(56), (23), (24)(56)$;
 $S_2: (143)(56), (1234)(56), (23)$;
 $H_1: \mathbb{Z}_2$;
 $\pi_1: \mathbb{Z}_2$.

APPENDIX B. THE MAGMA SCRIPT

```

// We first need to find, for each K^2, what are the possible baskets of
// singularities. By Lemma 1.8 the sum of the invariants B of the
// singularities must equal 3(8-K^2).
//
// We will represent a singular point 1/n(1,a) by the rational number
// a/n; hence a basket of singularities will be a multiset of rational
// numbers. Remember that cyclic quotient singularities 1/n(1,a) and
// 1/n(1,a') are isomorphic if a*a'=1 mod n, so we must consider rational
// numbers in (0,1) modulo the equivalence relation a/n~a'/n.
//
// The invariant B of a singularity 1/n(1,a) equals (a+a')/n+sum(b_i),
// where b_i are the entries of the continuous fraction of n/a: we see
// them as the sequence [b_1,...,b_r]. Note that the continuous
// fraction of n/a' is the "reversed" sequence [b_r,...,b_1].
//
// This can be seen as a bijection between rational numbers in (0,1)
// and sequences of integers strictly bigger than 1.
// We make this bijection explicit by the following scripts.

ContFrac:=function(s)
  CF:=[]; r:=1/s;
  while not IsIntegral(r) do
    Append(~CF, Ceiling(r)); r:=1/(C Ceiling(r)-r);
  end while;
  return Append(CF, r);
end function;

Nq:=func<cf|#cf eq 1 select cf[1] else cf[1]-1/$(Remove(cf,1))>;

RatNum:=func<seq|1/Nq(seq)>;

// "Wgt" computes the weight of a sequence, i.e., the sum of its
// entries. It bounds strictly from below B of the corresponding
// singular point.

```

```

Wgt:=function(seq)
  w:=0; for i in seq do w+=i; end for; return w;
end function;

// The next script computes all rational number whose continuous
// fraction has small weight, by listing all sequences (modulo
// "reverse") and storing the corresponding rational number.

RatNumsWithSmallWgt:=function(maxW)
  S:={ }; T:={ }; setnums:={RationalField()| };
  for i in [2..maxW] do Include(~S, [i]); end for;
  for i in [1..Floor(maxW/2)-1] do
    for seq in S do
      if #seq eq i then
        if maxW-Wgt(seq) ge 2 then
          for k in [2..maxW-Wgt(seq)] do
            Include(~S, Append(seq, k));
          end for; end if; end if;
        end for; end for;
      for seq in S do
        if Reverse(seq) notin T then Include(~T, seq);
        end if; end for;
      for seq in T do Include(~setnums, RatNum(seq)); end for;
    return setnums;
  end function;

// The next two scripts compute the invariants B and e of a rational
// number (i.e., of the corresponding singular point).

InvB:=func<r|Wgt(ContFrac(r))+r+RatNum(Reverse(ContFrac(r)))>;

Inve:=func<r|#ContFrac(r)+1-1/Denominator(RationalField()!r)>;

// The next two scripts compute the invariants B and e of a multiset
// of rational numbers (corresponding to a basket of singular points).

InvBSet:= function(basket)
  B:=0; for r in basket do B+=InvB(r); end for; return B;
end function;

InveSet:= function(basket)
  e:=0; for r in basket do e+=Inve(r); end for; return e;
end function;

// Here is the invariant k of the basket:

```



```

Invk:=func<r|InvBSet(r)-2*InveSet(r)>;

// The next script computes all rational numbers with weight bounded
// from above by maxW, as computed by RatNumsWithSmallWgt, and returns
// them in a sequence ordered by the value of their invariant B,
// starting from the one with biggest B.

OrderedRatNums:=function(maxW)
  seq:=[RationalField()| ]; seqB:=[RationalField()| ];
  set:=RatNumsWithSmallWgt(Floor(maxW));
  for r in set do i:=1;
    for s in seqB do
      if s gt InvB(r) then i+=1;
      else break s;
      end if; end for;
    Insert(~seq, i, r); Insert(~seqB, i, InvB(r));
  end for;
  return seq;
end function;

// The next one, CutSeqByB, takes a sequence "seq" and recursively
// removes the first element if its invariant B is at least maxB.

CutSeqByB:=function(seq,maxB)
  Seq:=seq;
  while #Seq ge 1 and InvB(Seq[1]) gt maxB do Remove(~Seq,1); end while;
  return Seq;
end function;

// Now we have a way to compute the set of rationals with B bounded by
// the integer maxB, ordered by B:
// CutSeqByB(OrderedRatNums(maxB-1),maxB)
//
// The next script takes a sequence of rationals ordered by B
// and computes the baskets with invariant exactly B that use only these
// rationals.
// The function is as follows:
// -- first remove the elements with B too big to be in a basket
// -- then take the first element, say r, if B(r)=B, store {* r *}
// -- else attach it to each basket with invariant B-B(r)
//    (computed recalling the function with the same sequence)
//    and store the result
// -- now we have all baskets containing r: remove r from the sequence
//    and repeat the procedure until the sequence is empty

```

```

BasketsWithSeqAndB:=function(seq,B)
  ratnums:=CutSeqByB(seq,B); baskets:={ };
  while #ratnums gt 0 do
    bigguy:=ratnums[1];
    if InvB(bigguy) eq B then Include(~baskets,{* bigguy *}); else
    for basket in $$ (ratnums, B-InvB(bigguy)) do
      Include(~baskets, Include(basket, bigguy));
    end for; end if;
    Remove(~ratnums,1);
  end while;
  return baskets;
end function;

// Now we can compute all Baskets with a given B:

BasketsWithSmallB:=func<B|
  BasketsWithSeqAndB(OrderedRatNums(Ceiling(B)-1),B)>;

// We do not need all these baskets, since most of them violate the Lemma 1.7.
// The next two scripts take care of this: "TestBasket" will check if a basket
// violates Lemma 1.7; "Basket" will take the output of BasketsWithSmallB and
// removes all the baskets which violate the condition.

TestBasket:=function(basket)
  firstseq=[];
  for r in basket do Append(~firstseq,r); end for;
  setofseqs:={ firstseq };
  for i in [1..#firstseq] do newseqs:={};
    for seq in setofseqs do
      Include(~newseqs,
        Insert(Remove(seq,i),i,RatNum(Reverse(ContFrac(seq[i])))));
    end for;
    setofseqs:=setofseqs join newseqs;
  end for;
  test:=false;
  for seq in setofseqs do
    if IsIntegral(Wgt(seq)) then test:=true;
    end if;
  end for;
  return test;
end function;

Baskets:=function(B)
  baskets:={ };
  for basket in BasketsWithSmallB(B) do
    if TestBasket(basket) then Include(~baskets, basket);
  end for;
end function;

```

```

    end if;
  end for;
  return baskets;
end function;

// Now we have found, for each  $K^2$ , a finite and rather small number of
// possible baskets. The next step is to restrict, for each basket, to finitely
// many signatures. We will represent a signature as the multiset of naturals
//  $\{m_i\}$ .
//
// We first define the index of a basket of singularities as the lowest
// common multiple of the indices of the singularities

GI:=func<r|Denominator(r)/GCD(Numerator(r)+1,Denominator(r))>;

GorInd:= function(bas)
  I:=1;
  for r in bas do I:=LCM(IntegerRing(!I,IntegerRing(!GI(r))); end for;
  return I;
end function;

// We need moreover the invariant Theta of a signature

Theta:=function(type)
  t:=-2; for n in type do t+=1-1/n; end for;
  return t;
end function;

// The input of the next program are 4 numbers, CardBasket, Length, SBound and
// HBound (SBound<=HBound), and its output are all signatures with
// #signature=Length such that (for  $C:=\max(1/6, (Length-3)/2)$ )
// 1) each  $m_i$  is smaller than HBound/C;
// 2) most  $m_i$  are smaller than SBound/C, the number of exceptions
//    being bounded from above by half of CardBasket.
// For sparing time, the script first checks if the length is smaller
// than the number of possible exceptions to 2, in which case only the
// inequality 1 is to consider.

CandTypes:=function(CardBasket,Length,SBound,HBound)
  C:=Maximum(1/6, (Length-3)/2); S:=Floor(SBound/C); H:=Floor(HBound/C);
  Exc:=Floor(CardBasket/2);
  if Length le Exc then Types:=Multisets({x: x in [2..H]},Length);
  else Types:=Multisets({x: x in [2..S]},Length);
    for k in [1..Exc] do
      for TypeBegin in Multisets({x: x in [2..S]},Length-k) do
        for TypeEnd in Multisets({x: x in [S+1..H]},k) do

```

```

    Include(~Types, TypeBegin join TypeEnd);
  end for; end for; end for;
end if;
return Types;
end function;

// The next script, ListOfTypesBas, finds all signatures compatible with the
// basket in the input (i.e., which respect Proposition 1.11).
// We use
// 1)  $\Theta \leq \max \Theta := (K^2 + k)/4$  (follows from 1.11.a),
// 2)  $\# \text{signature} \leq 2\Theta + 4$  (follows from the definition of  $\Theta$ ).

ListOfTypesBas:=function(basket)
  S:={ }; B:=InvBSet(basket); k:=Invk(basket); I:=GorInd(basket);
  Ksquare:=8-B/3; maxTh:=(Ksquare+k)/4;
  for h in [3..Floor(2*maxTh+4)] do
    for cand in CandTypes(#basket,h,maxTh+1,2*I*maxTh+1) do
      T:=Theta(cand);
      if T le maxTh then
        if T gt 0 then Alpha:=maxTh/T;
        if Alpha in IntegerRing() then
          if forall{n : n in cand | 2*Alpha*I/n in IntegerRing()} then bads:=0;
            for n in cand do
              if Alpha/n notin IntegerRing() then bads +=1;
            end if; end for;
          if bads le #basket/2 then Include(~S,cand);
          end if; end if; end if; end if; end if; end for; end for;
        return S;
      end function;

// Finally, we can conclude the second step, by writing a script which
// lists, for given  $K^2$ , all possible baskets (by using Baskets) and for
// each basket all possible signatures (by using ListOfTypesBas)

ListOfTypes:=function(Ksquare)
  S:=[* *];
  for basket in Baskets(3*(8-Ksquare)) do L:=ListOfTypesBas(basket);
    if not IsEmpty(L) then Append(~S,[* basket, L *]);
  end if; end for;
  return S;
end function;

// Now we are left with the last step: for each basket we need to
// consider all pairs of possible signatures and look for groups of the
// correct order which have two sets of spherical generators of these
// signatures which give a surface with the prescribed basket of

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// singularities. First we need to write some command which is not
// implemented in MAGMA.

// This extracts from a finite group the set of elements of a certain
// order.

ElsOfOrd:=function(group,order)
  Els:={ };
  for g in group do if Order(g) eq order then Include(~Els, g);
  end if; end for;
  return Els;
end function;

// TuplesOfGivenOrder creates a sequence of the same length as the input
// sequence seq, whose entries are subsets of the group in the input,
// and precisely the subsets of elements of order the corresponding
// entry of seq

TuplesOfGivenOrders:=function(group,seq)
  SEQ:=[];
  for i in [1..#seq] do
    if IsEmpty(ElsOfOrd(group,seq[i])) then SEQ:=[]; break i;
    else Append(~SEQ,ElsOfOrd(group,seq[i]));
    end if;
  end for;
  return SEQ;
end function;

// This two transform a multiset, resp. a tuple, into a sequence.

TypeToSeq:=function(type)
  seq:=[ ]; t:=type;
  while #t ne 0 do Append(~seq, Maximum(t));
    Exclude(~t, Maximum(t));
  end while;
  return seq;
end function;

TupleToSeq:=function(tuple)
  seq:=[];
  for elt in Tuplist(tuple) do
    Append(~seq,elt);
  end for;
  return seq;
end function;

```

```

// This script checks if a group has a set of spherical generators of
// the prescribed signature.

ExSphGens:=function(group,type)
  test:=false; seq:=TypeToSeq(type);
  SetCands:=TuplesOfGivenOrders(group,Prune(seq));
  if not IsEmpty(SetCands) then
    for cands in CartesianProduct(SetCands) do
      if Order(&*cands) eq seq[#seq] then
        if sub<group|TupleToSeq(cands)> eq group then
          test:=true; break cands;
        end if; end if;
      end for; end if;
    return test;
  end function;

// The next script runs a systematic search on all finite groups and
// produces the list of all triples (basket, pair of signatures, group)
// such that
// 1) the basket is compatible with the input  $K^2$ ;
// 2) the signatures are compatible with the basket;
// 3) the group has order  $(K^2+k)/(2*\Theta_1*\Theta_2)$  (see 1.11.b)
// and sets of spherical generators of both signatures.
// If one of the signatures is  $\{*2,3,7*\}$  the group must be perfect, so
// in this case the program first checks if there are perfect groups of
// the right order: if the answer is negative it jumps directly to the
// next case.
// The program skips to check the groups of order bigger than 2000, 1024
// (since there is no complete list available) or of orders in the set
// "badorders" which can be chosen by the user.
// These skipped cases are listed in the second output, and must be
// considered separately.

ListGroups:=function(Ksquare: badorders:={256,512,768,1152,1280,
                                           1536,1728,1792,1920})
  checked:=[* *]; tocheck:=[* *];
  for pair in ListOfTypes(Ksquare) do
    basket:=pair[1]; types:=pair[2]; k:=Invk(basket);
    for paioftypes in Multisets(types,2) do ord:=(Ksquare+k)/2;
      for T in paioftypes do ord:=ord/Theta(T);
      end for;
      if IsIntegral(ord) then
        if  $\{*2,3,7*\}$  in paioftypes and
          NumberOfGroups(PerfectGroupDatabase(),IntegerRing()!ord) eq 0
        then ;
        elif ord gt 2000 or ord in Include(badorders,1024) then

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    Append(~tocheck, [* basket, paioftypes, ord *]);
else for G in SmallGroups(IntegerRing()!ord: Warning := false) do
    test:=true;
    for T in paioftypes do
        if not ExSphGens(G,T) then test:=false; break T;
        end if;
    end for;
    if test then Append(~checked, [* basket, paioftypes, G *]);
    end if; end for;
end if; end if; end for; end for;
return checked, tocheck;
end function;

// Each case in the first output of ListGroups(K^2) gives at least a
// surface, but we are interested only in those surfaces having the
// prescribed basket of singularities. The next goal then is to compute
// these singularities.
//
// The next script takes a sequence of elements of a group and a further
// element g and conjugates each element of the sequence with g.

Conjug:=function(seq,elt)
    output:=[];
    for h in seq do Append(~output,h^elt);
    end for;
    return output;
end function;

// The next program computes all possible sets of spherical generators
// of a group of a prescribed signature and returns (to spare memory) only
// one of these sets for each conjugacy class.

SphGenUpToConj:=function(group,type)
    Set:={ }; Rep:={ }; seq:=TypeToSeq(type);
    SetCands:=TuplesOfGivenOrders(group,Prune(seq));
    if not IsEmpty(SetCands) then
        for cands in CartesianProduct(SetCands) do
            if TupleToSeq(cands) notin Set then
                if Order(&*cands) eq seq[#seq] then
                    if sub<group|TupleToSeq(cands)> eq group then
                        Include(~Rep, Append(TupleToSeq(cands),(&*cands)^-1));
                        for g in group do Include(~Set, Conjug(TupleToSeq(cands),g));
                        end for;
                    end if; end if; end if;
                end for; end if;
            end for; end if;
        end for; end if;
    end for; end if;
end function;

```

```

    return Rep;
end function;

// Given two sets of spherical generators, the singular points of the
// resulting surface are the image of points in the product of curves
//  $C_1 \times C_2$  having nontrivial stabilizer. These correspond to pairs
//  $(g_1, n_1, g_2, n_2)$  where
// -  $g_1$  is a generator of the first set;
// -  $g_2$  is a generator of the second set;
//  $1 \leq n_1 \leq \text{ord}(g_1)$ ;  $1 \leq n_2 \leq \text{ord}(g_2)$ ;  $g_1^{n_1} = g_2^{n_2}$ 
// First we write a program which computes the singular points
// coming from a fixed pair  $(g_1, g_2)$ .

BasketByAPairOfGens:=function(group,gen1,gen2)
    basket:={* *}; RC:={ }; delta:=GCD(Order(gen1),Order(gen2));
    alpha1:=IntegerRing()!(Order(gen1)/delta);
    alpha2:=IntegerRing()!(Order(gen2)/delta);
    RC2,f2:=RightTransversal(group,sub<group | gen2 >);
    for g2 in RC2 do test:=true;
        for g in sub<group | gen1 > do
            if f2(g2*g) in RC then test:=false; break g;
        end if; end for;
    if test then Include(~RC, g2);
    end if; end for;
    for g in RC do
        for d1 in [1..delta-1] do
            for d2 in [1..delta-1] do
                if (gen1^(d1*alpha1)) eq (gen2^(d2*alpha2))^g then
                    Include(~basket,d2/delta); break d1;
                end if; end for; end for; end for;
    return basket;
end function;

// We could use it to compute the basket of singularities of every
// constructed surface, but this is too expensive for our purposes.
// The next program only checks if, given two sets of spherical
// generators and a "candidate" basket, the resulting surface has the
// prescribed basket. The advantage is that in the wrong cases, the
// script stops when it finds a "forbidden" singularities, without
// losing time computing all the other singular points.

CheckSings:=function(basket,gens1,gens2,group)
    test:=true; bas:=basket;
    for gen1 in gens1 do
        for gen2 in gens2 do pb:=BasketByAPairOfGens(group,gen1,gen2);
            for r in pb do r1:=RatNum(Reverse(ContFrac(r)));

```



```

    if r in bas then Exclude(~bas,r);
    elif r1 in bas then Exclude(~bas,r1);
    else test:=false; break gen1;
    end if; end for;
  end for; end for;
  return test and IsEmpty(bas);
end function;

// The next script computes all product-quotient surfaces
// with  $p_g=0$ ,  $\chi=1$  and given  $K^2$ . It has the same input as ListGroups,
//  $K^2$  and the bad orders (B0), so it does not treat the cases not
// treated by ListGroups, which must be treated separately.

ExistingSurfaces:=function(Ksquare: B0:={256,512,768,1152,1280,
                                         1536,1728,1792,1920})
M:=[* *];
for triple in ListGroups(Ksquare: badorders:=B0) do
  basket:=triple[1]; pairsoftypes:=triple[2];
  group:=triple[3]; Types:=[];
  for type in pairsoftypes do Include(~Types,type); end for;
  SetGens1:=SphGenUpToConj(group,Types[1]);
  if #Types eq 1 then SetGens2:=SetGens1;
    else SetGens2:=SphGenUpToConj(group,Types[2]);
  end if;
  test:=false;
  for gens1 in SetGens1 do
    for gens2 in SetGens2 do
      if CheckSings(basket,gens1,gens2,group) then test:=true;
        break gens1;
      end if;
    end for; end for;
  if test then
    Append(~M, [* basket,pairsoftypes,IdentifyGroup(group)*]);
  end if;
end for;
return M;
end function;

// We still have not found all possible surfaces. In fact the output of
// ExistingSurfaces(n) gives all possible triples
// (basket,pair of signatures, group) which give AT LEAST a surface with
//  $p_g=0$  and  $K^2=n$ , but there could be more than one. In fact, there are
// more than one surface for each pair of spherical generators of the
// prescribed types which pass the singularity test, but they are often
// isomorphic. More precisely, they are isomorphic if the pair of
// spherical generators are equivalent for the equivalence relation

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// generated by Hurwitz moves (on each set of generators separately)
// and the automorphisms of the group (on both sets simultaneously).
// We need to construct orbits for this equivalence relation.
// The next scripts creates the Automorphism Group of a group as an
// explicit set.

AutGr:=
function(gr)
  Aut:=AutomorphismGroup(gr); A:={ Aut!1 };
  repeat
    for g1 in Generators(Aut) do
      for g2 in A do
        Include (~A,g1*g2);
      end for; end for;
    until #A eq #Aut;
  return A;
end function;

// The next script creates the Hurwitz move.

HurwitzMove:=
function(seq,idx)
  return Insert(Remove(seq,idx),idx+1,seq[idx]^seq[idx+1]);
end function;

// This script, starting from a sequence of elements of a group,
// creates all sequences of elements which are equivalent to the given
// one for the equivalence relation generated by Hurwitz moves,
// and returns (to spare memory) only the ones whose entries have never
// increasing order.

HurwitzOrbit:=
function(seq)
  orb:={ }; shortorb:={ }; Trash:={ seq };
  repeat
    ExtractRep(~Trash,~gens); Include(~orb, gens);
    for k in [1..#seq-1] do newgens:=HurwitzMove(gens,k);
    if newgens notin orb then Include(~Trash, newgens);
    end if; end for;
  until IsEmpty(Trash);
  for gens in orb do test:=true;
    for k in [1..#seq-1] do
      if Order(gens[k]) lt Order(gens[k+1]) then test:=false; break k;
    end if;
  end for;
  if test then Include(~shortorb, gens);

```

```

        end if;
    end for;
    return shortorb;
end function;

// Now we create all sets of spherical generators of a group of a
// prescribed signature.

SphGens:=function(group,seq)
    Gens:={ }; SetCands:=TuplesOfGivenOrders(group,Prune(seq));
    if not IsEmpty(SetCands) then
        for cands in CartesianProduct(SetCands) do
            if Order(&*cands) eq seq[#seq] then
                if sub<group|TupleToSeq(cands)> eq group then
                    Include(~Gens, cands);
                end if; end if;
            end for; end if;
        return Gens;
    end function;

// Finally, we can find all surfaces. The next program finds all
// surfaces with a given group, pair of signatures and basket (must be run
// on the outputs of ExistingSurfaces).

FindSurfaces:=function(basket, paioftypes, gr)
    Good:={@ @}; Surfaces:={ }; All:={ }; Aut:=AutGr(gr); Types:=[];
    for type in paioftypes do Append(~Types, type);
    end for;
    seq1:=TypeToSeq(Types[1]); seq2:=TypeToSeq(Types[2]);
    NumberOfCands:=#SphGens(gr,seq1)*#SphGens(gr,seq2);
    for gens1 in SphGens(gr,seq1) do genseq1:=TupleToSeq(gens1);
    for gens2 in SphGens(gr,seq2) do genseq2:=TupleToSeq(gens2);
        if genseq1 cat genseq2 notin All then
            Include(~Surfaces, [Append(genseq1,(&*gens1)^-1),
                                Append(genseq2,(&*gens2)^-1)]);
            orb1:=HurwitzOrbit(Append(genseq1,(&*gens1)^-1));
            orb2:=HurwitzOrbit(Append(genseq2,(&*gens2)^-1));
            for g1 in orb1 do gg1:=Prune(g1);
            for g2 in orb2 do gg2:=Prune(g2);
                if gg1 cat gg2 notin All then
                    for phi in Aut do Include(~All, phi(gg1 cat gg2));
                    end for;
                end if;
            if #All eq NumberOfCands then break gens1;
            end if;
        end for; end for;
end function;

```

```

        end if;
    end for; end for;
    for gens in Surfaces do
    if CheckSings(basket,gens[1],gens[2],gr) then
        Include(~Good, gens);
    end if; end for;
    return Good;
end function;

// The next script, FindCurves, uses the same argument of FindSurfaces to find
// all curves with a given signature and group, modulo Hurwitz moves and inner
// automorphisms of the group.

FindCurves:=function(type, gr)
    Curves:={ }; All:={ }; seq:=TypeToSeq(type);
    NumberOfCands:=#SphGens(gr,seq);
    for gens in SphGens(gr,seq) do genseq:=TupleToSeq(gens);
        if genseq notin All then
            Include(~Curves, Append(genseq,(&*gens)^-1));
            orb:=HurwitzOrbit(Append(genseq,(&*gens)^-1));
            for g in orb do gg:=Prune(g);
                if gg notin All then
                    for h in gr do Include(~All, Conjug(gg,h));
                end for;
            end if;
            if #All eq NumberOfCands then break gens;
        end if;
    end for;
    return Curves;
end function;

PolyGroup:=function(seq,gr)
    F:=FreeGroup(#seq); R:={F![1..#seq]};
    for i in [1..#seq] do
        Include(~R,F.i^Order(seq[i]));
    end for;
    P:=quo<F|R>;
    return P, hom<P->gr|seq>;
end function;

DirProd:=function(G1,G2)
    G1xG2:=DirectProduct(G1,G2); vars:=[];
    n:=[NumberOfGenerators(G1),NumberOfGenerators(G2)];
    for i in [1..Wgt(n)] do Append(~vars,G1xG2.i); end for;

```

```

SplittedVars:=Partition(vars,n);
injs:=[hom< G1->G1xG2 | SplittedVars[1]>,
      hom< G2->G1xG2 | SplittedVars[2]>];
vars1:=[]; vars2:=[];
for i in [1..n[1]] do
  Append(~vars1,G1.i); Append(~vars2,G2!1);
end for;
for i in [1..n[2]] do
  Append(~vars1,G1!1); Append(~vars2,G2.i);
end for;
projs:=[hom< G1xG2->G1 | vars1>,hom< G1xG2->G2 | vars2>];
return G1xG2, injs, projs;
end function;

// The next script computes, given two maps A->B (careful, they MUST be
// between the same groups) the map product induced by the product on B

MapProd:=function(map1,map2)
  seq:=[]; G:=Domain(map1); H:=Codomain(map1);
  if Category(G) eq GrpPC then n:=NPCgens(G);
  else n:=NumberOfGenerators(G); end if;
  for i in [1..n] do Append(~seq,map1(G.i)*map2(G.i)); end for;
  return hom<G->H|seq>;
end function;

// Finally, this program computes the fundamental group of a product-quotient
// surface.

Pi1:=function(pairsofseqs,gr)
  T1,f1:=PolyGroup(pairsofseqs[1],gr);
  T2,f2:=PolyGroup(pairsofseqs[2],gr);
  T1xT2,inT,proT:=DirProd(T1,T2);
  grxgr,inG:=DirectProduct(gr,gr);
  Diag:=MapProd(inG[1],inG[2])(gr);
  f:=MapProd(proT[1]*f1*inG[1],proT[2]*f2*inG[2]);
  H:=Rewrite(T1xT2,Diag@@f); rels:=[];
  for i in [1..#pairsofseqs[1]] do g1:=pairsofseqs[1][i];
  for j in [1..#pairsofseqs[2]] do g2:=pairsofseqs[2][j];
  for d1 in [1..Order(g1)-1] do
  for d2 in [1..Order(g2)-1] do
    test,h:=IsConjugate(gr,g1^d1,g2^d2);
    if test then for c in Centralizer(gr,g1^d1) do
      Append(~rels, T1xT2.i^d1 *
        ((T1xT2.(j+#pairsofseqs[1])^d2)^(inT[2]((h^-1*c) @@ f2))));
    end for; end if;
  end for; end for; end for; end for;
end function;

```

```
    return Simplify(quo<H|rels>);  
end function;
```